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A Theory of Commodity Price Fluctuations
Technical Appendix

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The following pages constitute the technical appendix to *A Theory of Commodity Price Fluctuations,* University of Essex Discussion Paper, 432, August 1994. The revised version of the paper, excluding the technical appendix, appears in the *Journal of Political Economy,* volume 104, October 1996. Minor changes of terminology have been made in the appendix for consistency with the published version of the paper.

Technical Appendix

The appendix begins with a statement, without proof, of **Theorem 1** which characterizes the equilibrium price function in the case of i.i.d. disturbances. The building blocks for the existence theorem (**Theorem 2**) with time dependent disturbances are contained in **lemmas 1–3**. The proof of theorem 2 is followed by **lemma 4** and **Theorem 3** which establishes a property of the equilibrium price function for positively autocorrelated disturbances (*Assumption* 4a). The building blocks for the periodic disturbances model are contained in **lemmas 5–7**, resulting in **Theorem 4** which characterizes the equilibrium price functions for epochs each of which comprises exactly two primitive periods. Finally, **Theorem 4′** provides price functions appropriate for a generalization of the periodic disturbances assumption to a multi-period setting.

Assumption 1.

(i) The demand function, $D: (p_0, p_1) \rightarrow \mathbf{R}$ is continuous and strictly decreasing on its domain with $0 \leq p_0 < p_1 < \infty$ and $\lim_{p \rightarrow p_0} D(p) = +\infty$.

(ii) $0 < P(w_*) < p_1$, where w_* denotes the infimum of the support for the disturbance, w.

Assumption 2: $0 < \theta \equiv (1 - \delta)/(1 + r) < 1$,

The random process generating the shocks ('harvest fluctuations') is expressed by the *transition function* $Q^s(w, w')$ which is interpreted as the conditional probability of next period's harvest, w', given the observation w of the current period's harvest. That is,

$$Q^s(a, A) = \Pr\{w_{t+1} \in A \,|\, w_t = a\} \tag{A.1}$$

is the probability that the harvest in period $t+1$ is a member of the set A, given that the harvest in period t is equal to a. Each harvest, w, is assumed to be a member of a set,

$$W^s = \{w^s \in \mathbf{R} \,|\, -\infty < \underline{w}^s \leq w^s \leq \overline{w}^s < +\infty\} \tag{A.2}$$

and $(W^s, \mathcal{W}^s)$ is a measurable space, where $\mathcal{W}^s$ is the Borel algebra of sets for W^s. The superscript, s, indicates that the probability distributions of disturbances may, in the most general case, differ across time periods.

Assumption 3a (i.i.d. disturbances). The disturbances w are identically and independently distributed with compact support:

$$W = \{w \in \mathbf{R} \,|\, -\infty < \underline{w} \leq w \leq \overline{w} < +\infty\}. \tag{A.3}$$

Also, $w_* = \underline{w}$ (see assumption 1).

Theorem 1. *Under assumptions 1, 2, 3a there exists a unique price function* $f : X \rightarrow \mathbf{R}$, *where* $X = \{x \,|\, x \in \mathbf{R},\ x \geq \underline{w}\}$, *such that* $f(\cdot)$ *is continuous, non-negative* *and non-increasing and satisfies:*

$$f(x) = \max\left[\theta\int_W f[w' + (1-\delta)(x - D(f(x)))]Q(dw'),\, P(x)\right]. \tag{A.4}$$

Proof. See Deaton and Laroque (1992). ‖

TIME DEPENDENT DISTURBANCES.

In this case the time superscripts may be omitted, so that:

$$W = \{w \in \mathbf{R} \mid -\infty < \underline{w} \le w \le \overline{w} < +\infty\} \tag{A.5}$$

Once again, $w_* = \underline{w}$ (see assumption 1).

Definition. Given the measurable space $(W, \mathcal{W})$, the *transition function,* $Q : W \times \mathcal{W} \to [0,\, 1]$ is such that:

(i) for each $w \in W$, $Q(w, \cdot)$ is a probability measure on $(W, \mathcal{W})$;

(ii) for each $A \in \mathcal{W}$, $Q(\cdot, A)$ is a $\mathcal{W}$-measurable function.

Assumption 3b. (Time dependent disturbances)

The transition function Q defined on $W \times \mathcal{W}$ (where $\mathcal{W}$ denotes the Borel algebra of sets for W) has the following properties:

(i) For any continuous function $h : (W \times W) \to \mathbf{R}$, which is bounded (with respect to the Euclidean metric), the operator J defined by

$$(Jh)(w) = \int_W h(w,\, w')Q(w,\, dw'), \quad \text{for each } w \in W \tag{A.6}$$

is such that $(Jh) : W \to \mathbf{R}$ is a continuous, bounded function.[1]

(ii) $Q(w, A)$ is continuous in $w \in W$ for each $A \in \mathcal{W}$.

Consider the functions f, g and associated mapping, $T : g \to f$ defined, for each $w \in W$, by:

$$f(y,\, w) = \max\left[\theta\int_W g[(1-\delta)(y + w - D(f(y,\, w))),\, w']Q(w,\, dw'),\, P(y+w)\right]. \tag{A.7}$$

In this case the Stationary Rational Expectations Equilibrium, *SREE,* which is sought takes the form of a function $g(y, w)$ such that $f(y, w) = g(y, w)$ for each $w \in W$, that is, $f = Tf$.

The function g is chosen to belong to the space $\mathbf{G}$ characterized by:

Definition. $\mathbf{G}$ is the space of functions $g(y, w)$ such that:

[1] This requirement is an extension of the *Feller property* applied to transition functions. See Stokey and Lucas (1989), p.220.

(i) The domain of $g(y, w)$ is $\Lambda = (Y \times W)$ where $Y = \{y \,|\, y \in \mathbf{R},\ y \geq 0\}$.

(ii) $g(y, w)$ is continuous, non-negative, non-increasing in y and is continuous in w.

(iii) $g(0, \cdot) \leq P(\underline{w})$, that is, the members of $\mathbf{G}$ are bounded above by the value $P(\underline{w})$ for every $(y, w) \in \Lambda$.

Definition. The metric $d(g_i, g_j),\ g_i, g_j \in \mathbf{G}$ is defined by

$$d(g_i, g_j) = \sup_{(y, w) \in \Lambda} |g_i(y, w) - g_j(y, w)|.$$

The properties of $\mathbf{G}$ (in particular the boundedness of every $g \in \mathbf{G}$) guarantee that the metric d is well-defined.

The subsequent presentation is simplified by introducing a function G with domain $\Gamma \times W$, where $\Gamma = \{(p, y, w) \,|\, P(y + w) \leq p < p_1,\ y \in Y,\ w \in W\}$, defined by:

$$G(q, y, w) = \theta \int_W g[(1 - \delta)(y + w - D(q)), w']Q(w, dw'). \tag{A.8}$$

Lemma 1: *T maps $\mathbf{G}$ into itself.*

Proof. For any given $g \in \mathbf{G}$, the proof shows that $f = Tg \in \mathbf{G}$ by examining the properties of the G function defined in equation (A.8).

That the function G is continuous in q, y, w, may be shown by an argument similar to that of Stokey and Lucas (1989), pp.261–2. Choose a sequence $(q_n, y_n, w_n) \rightarrow (q, y, w)$. It follows that

$$\begin{aligned} |G(q, y, w) - G(q_n, y_n, w_n)| &\leq |G(q, y, w) - G(q, y, w_n)| + |G(q, y, w_n) - G(q_n, y_n, w_n)| \\ &\leq |G(q, y, w) - G(q, y, w_n)| + H_n \end{aligned} \tag{A.9}$$

where

$$H_n \equiv \theta \int_W |g[(1-\delta)(y + w - D(q)), w'] - g[(1-\delta)(y_n + w_n - D(q_n)), w']|Q(w_n, dw').$$

Given that g is bounded and continuous by construction, the first term in (A.9) tends to zero as $n \rightarrow \infty$. Also since $(q_n, y_n, w_n) \rightarrow (q, y, w)$ as $n \rightarrow \infty$, there exists a compact set, $K \subset \Gamma$ such that $(q, y, w) \in \Gamma$ and $(q_n, y_n, w_n) \in K$, for all n. Because g is continuous, it is uniformly continuous on the compact set K. That is, for all $\varepsilon > 0$ there is an $N \geq 1$ such that

$$|g[(1-\delta)(y + w - D(q)), w'] - g[(1-\delta)(y_n + w_n - D(q_n)), w']| < \varepsilon,\ \forall n > N,\ w' \in W.$$

Hence $H_n \rightarrow 0$ as $n \rightarrow \infty$ thus demonstrating the continuity of G.

For any $(y,\, w) \in \Lambda$, $f(y,\, w)$ solves

$$\max_q [G(q,\, y,\, w) \,-\, q,\;\; P(y+w) \,-\, q] = 0. \tag{A.10}$$

The properties of the solution to (A.10) follow from the properties of $G(q,\, y,\, w)$. It has already been shown that G is continuous. Since g is non-negative and non-increasing in y, it follows that G is continuous, non-negative and non-increasing in q, y. Moreover, $G \,-\, q$ is strictly decreasing in q.

Note from (A.10) that its solution must satisfy $q \,\geq\, P(y+w)$. Suppose that $q \,>\, P(y+w)$; then, since g is non-increasing in y, $G(q,\, y,\, w)$ is non-increasing in q and,

$$\begin{aligned} G(q,\, y,\, w) \;&\leq\; G(P(y+w),\, y,\, w) \;=\; \theta \int_W g(0,\, w')Q(w,\, dw') \\ &\leq\; \theta \int_W P(\underline{w})Q(w,\, dw') \\ &=\; \theta P(\underline{w}) \;\leq\; \theta p_1 \;<\; p_1 \end{aligned} \tag{A.11}$$

so that $G(q,\, y,\, w) \,-\, q$ tends to a negative number as $q \,\rightarrow\, p_1$.

At $q \,=\, P(y+w)$, if $G(q,\, y,\, w) - q = G(P(y+w),\, y,\, w) \,-\, P(y+w) \,\leq\, 0$, it follows from the monotonicity of G in q that $G(q,\, y,\, w) \,-\, q$ is negative for all $q \,>\, P(y+w)$ and the solution to (A.10) must be $f(y,\, w) \,=\, P(y+w)$.

Alternatively, if at $q \,=\, P(y+w)$, $G(q,\, y,\, w) - q = G(P(y+w),\, y,\, w) - P(y+w) \,>\, 0$, it follows, again from the monotonicity of G in q, that $G(q,\, y,\, w) \,-\, q$ has a unique zero root, $q \,=\, f(y,\, w)$ such that $G(f(y,\, w),\, y,\, w) \,-\, f(y,\, w) \equiv\, 0$.

Thus the solution to (A.10) provides a function $f(y,\, w)$ which is unique, continuous and non-negative for $(y,\, w) \,\in\, \Lambda$ and non-increasing in $y \,\in\, Y$.

It remains to show that, if $g(0,\, w) \,\leq\, P(\underline{w})$, then $f(0,\, w) \,\leq\, P(\underline{w})$ for any $w \,\in\, W$. Now, either (a) $f(0,\, w) \,=\, P(w)$; or, (b) $f(0,\, w) \,=\, G(f(0,\, w),\, 0,\, w)$.

In case (a):

$$\begin{aligned} f(0,\, w) \;&=\; P(w) \\ &\leq\; P(\underline{w}) \quad \text{since } P(\cdot) \text{ is decreasing, and } w \,\geq\, \underline{w}. \end{aligned}$$

In case (b):

$$\begin{aligned} f(0,\, w) \;&=\; \theta \int_W g[(1-\delta)(w - D(f(0,w)),\, w']Q(w,\, dw') \\ &\leq\; \theta \int_W g[0,\, w']Q(w,\, dw') \quad \text{since the net carryover is non-negative} \\ &\qquad\qquad\qquad\qquad\qquad\quad \text{and } g(\cdot,\, w) \text{ is non-increasing,} \\ &\leq\; \theta P(\underline{w}) \quad \text{since } g(0,\, \cdot) \leq P(\underline{w}), \\ &<\; P(\underline{w}) \quad \text{since } 0 \,<\, \theta \,<\, 1. \end{aligned} \tag{A.12}$$

Thus $f = Tg \in \mathbf{G}$, as asserted. $\|$

Lemma 2: T is a contraction mapping.

Proof. Choose any pair $g_0,\, g_1 \in \mathbf{G}$ such that

$$g_1(y,\, w) \leq g_0(y,\, w) \leq g_1(y,\, w) + a, \quad \forall\, (y,\, w) \in \Lambda \tag{A.13}$$

where a is a positive constant. It can now be shown that the mapping T preserves the inequalities.

Note, first, that $G_1 \leq G_0$ from the definition of G (where the subscripts correspond to different g functions but evaluated at the same q, y and w).

Fix $(y,\, w) \in \Lambda$ and let q_0 denote the (unique) root of $G_0(q,\, y,\, w) - q = 0$, if it exists. Since $G_1(q,\, y,\, w) \leq G_0(q,\, y,\, w)$, then $G_1(q_0,\, y,\, w) - q_0 \leq 0$. Hence the root, q_1, of $G_1(q,\, y,\, w) - q = 0$ must satisfy $P(y+w) \leq q_1 \leq q_0$, thus establishing that $Tg_1 \leq Tg_0$. Trivially, if either or both of the roots does not exist, the solution will be $q_0 = P(y+w)$ or $q_1 = P(y+w)$, so that it is just possible that $Tg_1 = Tg_0$.

Now consider $T(g_1 + a)$,

$$\begin{aligned} T(g_1 + a) &= \max[G(g_2(y,\, w),\, y,\, w) + a,\, P(y+w)] \\ &\leq \theta a + \max[G(g_2(y,\, w),\, y,\, w),\, P(y+w)] \\ &= \theta a + Tg_1 \end{aligned} \tag{A.14}$$

Thus,

$$Tg_1(y,\, w) \leq Tg_0(y,\, w) \leq Tg_1(y,\, w) + \theta a. \tag{A.15}$$

Now set $a = d(g_0,\, g_1)$ using the metric defined above. Then,

$$\begin{aligned} Tg_0 - Tg_1 &\leq \theta a \\ |Tg_0 - Tg_1| &\leq \theta a \quad \text{for every } (y,\, w) \in \Lambda \\ d(Tg_0,\, Tg_1) &\leq \theta a = \theta d(g_0,\, g_1) \end{aligned} \tag{A.16}$$

establishing that T is a contraction mapping. $\|$

Lemma 3: The space $\mathbf{G}$ is complete in the metric d.

Proof. The assertion of this lemma is a well known result for sets of functions with the properties of $\mathbf{G}$ and which are also normed vector spaces. (See for example, Stokey and Lucas (1989), p.47.) Because $\mathbf{G}$ is not a vector space of functions, it is necessary to check that it is complete. That this is straightforward may be verified from inspection of the proof of Theorem 3.1 in Stokey and Lucas. The required modification of their argument is given below.

For notational simplicity, denote $\lambda,\ \kappa\ \in\ \mathbf{R}^2$, each vector being intended to represent a member $(y,\ w)\ \in\ \Lambda$, defined above. The space $(\mathbf{G},\ d)$ is complete if every Cauchy sequence $\{g_n\}_{n=0}^{\infty}$ with $g_n\ \in\ \mathbf{G}$ converges to a function $g\ \in\ \mathbf{G}$ for all $\lambda\ \in\ \Lambda$.

First, it is necessary to construct a limit function g corresponding to each Cauchy sequence. By the definition of a Cauchy sequence, for any $\varepsilon\ >\ 0$ there is an N_ε such that $d(g_r,\ g_s)\ <\ \varepsilon$ for every $r,\ s\ >\ N_\varepsilon$. For a given, arbitrary $\lambda\ \in\ \Lambda$,

$$|g_r(\lambda)\ -\ g_s(\lambda)|\ \leq\ \sup_{\lambda\in\Lambda}|g_r(\kappa)\ -\ g_s(\kappa)|\ =\ d(g_r,\ g_s)\ <\ \varepsilon\quad \text{for } r,\ s\ >\ N_\varepsilon$$

so that the sequence of real numbers, $\{g_n(\lambda)\}$, satisfies the Cauchy criterion. From the completeness of $\mathbf{R}$, the limit point $g(\lambda)\ \in\ \mathbf{R}$, thus providing the required function g.

Second, it is necessary to show that g is the limit point of $\{g_n\}$ using the metric d. For any $\varepsilon\ >\ 0$, it is possible to choose N_ε such that $d(g_r,\ g_s)\ <\ \varepsilon/2$ for all $r,\ s\ >\ N_\varepsilon$, because $\{g_n\}$ is a Cauchy sequence. Fix $\lambda\ \in\ \Lambda$,

$$\begin{aligned}|g_r(\lambda)-g(\lambda)| &\leq |g_r(\lambda)-g_s(\lambda)|+|g_s(\lambda)-g(\lambda)|\\ &\leq d(g_r,\ g_s)+|g_s(\lambda)-g(\lambda)|\\ &< \varepsilon/2+|g_s(\lambda)-g(\lambda)|. \qquad \text{(A.17)}\end{aligned}$$

By construction the sequence $\{g_n(\lambda)\}$ converges to $g(\lambda)$ so that s can be chosen for the fixed λ such that $|g_s(\lambda)\ -\ g(\lambda)|\ <\ \varepsilon/2$. Hence, $d(g_r,\ g)\ <\ \varepsilon$ for every $r\ \geq\ N_\varepsilon$. Given that any $\varepsilon\ >\ 0$ can be chosen, it follows that g is the limit point of the sequence.

Thirdly, it must be demonstrated that g is continuous, that is, for all $\lambda\ \in\ \Lambda$ and any $\varepsilon\ >\ 0$ there exists a $\delta\ >\ 0$ such that $|g(\lambda)\ -\ g(\kappa)|\ <\ \varepsilon$ whenever $\|\lambda\ -\ \kappa\|_E\ <\ \delta$ where $\|\cdot\|_E$ denotes the Euclidean norm in $\mathbf{R}^2$. Fix λ and ε and choose g_s such that $d(g,\ g_s)\ <\ \varepsilon/3$.

Now, since g_s is continuous, it is possible to choose δ such that $\|\lambda\ -\ \kappa\|_E\ <\ \delta$ implies $|g_s(\lambda)\ -\ g_s(\kappa)|\ <\ \varepsilon/3$. Hence,

$$\begin{aligned}|g(\lambda)-g(\kappa)| &\leq |g(\lambda)-g_s(\lambda)|+|g_s(\lambda)-g_s(\kappa)|+|g_s(\kappa)-g(\kappa)| \qquad \text{(A.18)}\\ &< d(g,\ g_s)+|g_s(\lambda)-g_s(\kappa)|+d(g,\ g_s) \qquad \text{(A.19)}\\ &< \varepsilon \qquad \text{(A.20)}\end{aligned}$$

as required.

Finally, that the limit point g is non-increasing, non-negative and satisfies $g(0,\ \cdot)\ \leq\ P(\underline{w})$ follows from the continuity of g and the convergence of the $\{g_n\}$ sequence for every $\lambda\ \in\ \Lambda$. $\|$

Theorem 2. *Under assumptions 1, 2, 3b there exists a unique price function $f : \Lambda \to \mathbf{R}$, such that $f(\cdot, \cdot)$ is continuous, non-negative and non-increasing in its first argument and satisfies:*

$$f(y, w) = \max\left[\theta \int_W f[(1-\delta)(y + w - D(f(y, w))], w')Q(w, dw'), P(y + w)\right]. \tag{A.21}$$

Proof. The proof demonstrates that T has a unique fixed point.

Choose any $g_0 \in \mathbf{G}$ and consider the sequence:

$$g_0, \quad g_1 = Tg_0, \quad g_2 = Tg_1, \quad \ldots, \quad g_t = Tg_{t-1}, \quad \ldots$$

The definition of d implies that $d(g_{t-1}, g_t) \leq \theta^{t-1} d(g_0, g_1)$. Hence, from assumption 2, the sequence converges uniformly. Denote the limit of the sequence by f. Then from lemma 1, $f \in \mathbf{G}$.

To show that the limit is an equilibrium, that is, $Tf = f$, consider

$$\begin{aligned} d(Tf, f) &\leq d(Tf, T^n g_0) + d(T^n g_0, f) \\ &\leq \theta d(f, T^{n-1} g_0) + d(T^n g_0, f) \quad \forall\, n,\ g_0 \in \mathbf{G} \end{aligned} \tag{A.22}$$

Both terms on the right hand side of the inequality tend to zero as $n \to \infty$, so that $d(Tf, f) = 0$ and f is an equilibrium.

To show that the equilibrium is unique, suppose that $\tilde{f} \in \mathbf{G}$ is another solution $T\tilde{f} = \tilde{f}$. Then,

$$0 < \alpha = d(\tilde{f}, f) = d(T\tilde{f}, Tf) \leq \theta d(\tilde{f}, f) = \theta\alpha$$

which is a contradiction. Hence f is unique. ||

Assumption 4a. (Positive autocorrelation)
For any function $h(w')$ which is non-increasing (*resp.* non-decreasing) in w', the transition function, $Q(w, w')$, is such that

$$\int_W h(w')Q(w_1, dw') \leq \text{ (resp. } \geq) \int_W h(w')Q(w_2, dw') \tag{A.23}$$

for all $w_1 > w_2$ such that $w_1, w_2 \in W$.

Lemma 4. *Let $T : \mathbf{G} \to \mathbf{G}$ be a contraction mapping with $f = Tf \in \mathbf{G}$. If $\mathbf{G}' \subseteq \mathbf{G}$ is closed and if $T(\mathbf{G}') \subseteq \mathbf{G}'$, then $f \in \mathbf{G}'$.*

Proof. See Stokey and Lucas (1989), p.52. The proof shows that, starting from any function in $\mathbf{G}'$, the repeated application of the operator T generates a convergent sequence of functions in $\mathbf{G}'$ (by hypothesis). From the assumption that $\mathbf{G}'$ is closed in the metric d, it follows that the limit point of the sequence is also a member of $\mathbf{G}'$ as asserted. ||

Theorem 3. *Let $f(y, w)$ denote the unique SREE defined in expression (A.21). Then, under assumptions 1, 2, 3b, 4a, for each $y \in Y$, $f(y, \cdot) : W \to \mathbf{R}$ is non-increasing.*

Proof.

Let $\mathbf{G}' \subseteq \mathbf{G}$ be the set of functions in $\mathbf{G}$ which are non-increasing in w. From lemma 4, it suffices to show that for any $g \in \mathbf{G}'$ it follows that $Tg \in \mathbf{G}'$. To see that this is indeed the case, let $g \in \mathbf{G}'$ and choose $\overline{w} \geq w_1 > w_2 \geq \underline{w}$. Define the function $G(q, y, w)$ corresponding to $g(y, w)$ as in expression (A.8), above.

Thus,

$$G(q, y, w_1) = \theta \int_W g[(1-\delta)(w_1 + y - D(q)), w']Q(w_1, dw') \tag{A.24}$$

$$G(q, y, w_2) = \theta \int_W g[(1-\delta)(w_2 + y - D(q)), w']Q(w_2, dw') \tag{A.25}$$

In order to apply assumption 4a, construct the function $\tilde{G}(q, y, w_1, w_2)$ as follows:

$$\tilde{G}(q, y, w_1, w_2) = \theta \int_W g[(1-\delta)(w_2 + y - D(q)), w']Q(w_1, dw'). \tag{A.26}$$

Now interpret $g(\cdot, w')$ in the integrand of (A.25) and (A.26) with given y, q, w_2 as $h(w')$ in Assumption 4a. It follows immediately from the fact that $g(\cdot, \cdot)$ is non-increasing in its first argument, and assumption 4a, that

$$G(q, y, w_1) \leq \tilde{G}(q, y, w_1, w_2) \leq G(q, y, w_2). \tag{A.27}$$

Now, using the method of lemma 1, application of the transformation $f = Tg$ implies that $f(y, w)$ solves

$$\max_q [G(q, y, w) - q, \; P(y+w) - q] = 0 \tag{A.28}$$

for each value of w and, in particular, for w_1 and w_2.

From the argument in lemma 1, it follows that the solution to (A.28) with w_1 and w_2, respectively, must satisfy exactly one of the following four conditions:

(a) $f(y, w_1) = P(y + w_1)$ and $f(y, w_2) = P(y + w_2)$

(b) $f(y, w_1) = P(y + w_1)$ and $f(y, w_2) = G(f(y, w_2), y, w_2)$.

(c) $f(y, w_1) = G(f(y, w_1), y, w_1)$ and $f(y, w_2) = P(y + w_2)$.

(d) $f(y, w_1) = G(f(y, w_1), y, w_1)$ and $f(y, w_2) = G(f(y, w_2), y, w_2)$.

In case (a), $f \in \mathbf{G}'$, since $P(\cdot)$ is a decreasing function. The functions in case (d) arise if $G(P(y + w_i), x, w_i) - P(y + w_i) > 0$ for $i = 1, 2$, so that $G(q, y, w_1) - q$ and $G(q, y, w_2) - q$ have (unique) zero roots for q, $q_1 = f(y, w_1)$ and $q_2 = f(y, w_2)$, respectively. Since $G(q, y, w)$ is non-increasing in w it follows that $f(y, w_1) = q_1 \leq q_2 = f(y, w_2)$, and $f \in \mathbf{G}'$, as asserted.

The intermediate case (b) occurs if $G(P(y+w_1),\, y,\, w_1) \,\leq\, P(y+w_1)$ and $G(P(y+w_2),\, y,\, w_2) \,>\, P(y+w_2)$ so that:

$$f(y,\, w_1) \;=\; P(y+w_1) \;<\; P(y+w_2) \;<\; G(P(y+w_2),\, y,\, w_2) \;\leq\; f(y,\, w_2) \qquad \text{(A.29)}$$

so that $f \,\in\, \mathbf{G}'$, as asserted.

The intermediate case (c) occurs if $G(P(y+w_2),\, y,\, w_2) \,\leq\, P(y+w_2)$ and $G(P(y+w_1),\, y,\, w_1) \,>\, P(y+w_1)$. Thus, if q_1 and q_2 are the respective roots of $G(q,\, y,\, w_1) - q$ and $G(q,\, y,\, w_2) \,-\, q$, then $q_1 \,\leq\, G(q_1,\, y,\, w_2)$, since $G(q,\, y,\, w) \,-\, q$ is non-increasing in w. Also $q_2 \,\geq\, q_1$, since $G(q,\, y,\, w) \,-\, q$ is decreasing in q. Thus:

$$f(y,\, w_1) \;=\; q_1 \;\leq\; q_2 \;\leq\; P(y+w_2) \;=\; f(y,\, w_2). \qquad \text{(A.30)}$$

so that $f \,\in\, \mathbf{G}'$, as asserted.

For each case the argument establishes that $(Tg)(y,\, \cdot) \,\in\, \mathbf{G}'$ for each $y \,\in\, Y$ thus demonstrating that $f(y,\, \cdot) : W \,\rightarrow\, \mathbf{R}$ is a non-increasing function. $\|$

Periodic Disturbances

In this case the disturbances are assumed to be temporally independent but to come from heterogeneous distributions, thus providing what is called the 'periodic disturbances' model. For notational simplicity, the bulk of the analysis below is for the case of epochs comprising two periods. The generalization to multi-period epochs is straightforward and forms theorem $4'$.

The periods are named as *even,* e or *odd,* o. Each harvest, w^k, is assumed to be a member of a set,

$$W^k \;=\; \{w^k \,\in\, \mathbf{R} \,|\, -\infty \,<\, \underline{w}^k \,\leq\, w^k \,\leq\, \overline{w}^k \,<\, +\infty\} \qquad \text{(A.31)}$$

and $(W^k,\, \mathcal{W}^k)$ is a measurable space, where $\mathcal{W}^k$ is the Borel algebra of sets for W^k and where $k \,=\, e,\, o$.

Assumption 3c (periodic disturbances). The disturbances w^o, w^e are independently distributed. That is, the functions $Q^k : \mathcal{W}^k \,\rightarrow\, [0,\, 1]$ are probability measures on $\mathcal{W}^k$ for $k \,=\, e,\, o$, respectively.

The lower bound, w_*, in assumption 1 now becomes $w_* \,=\, \min[\underline{w}^e,\, \underline{w}^o]$, and assumption 2 remains unchanged.

Consider the relationship and associated mapping, T^h defined by:

$$f^h(x) \;=\; \max\left[\theta \int_{W^k} g^k[w' + (1-\delta)(x - D(f^h(x)))]Q^k(dw'),\, P(x)\right], \qquad T^h \,:\, g^k \,\rightarrow\, f^h \qquad \text{(A.32)}$$

for $h,\, k \,=\, e,\, o$ and $h \,\neq\, k$. That is, $T^o \,:\, g^e \,\rightarrow\, f^o$ and $T^e \,:\, g^o \,\rightarrow\, f^e$.

Define the composite mappings S^e and S^o by:

$$S^e \;=\; T^e \cdot T^o : \; g^e \rightarrow f^e \tag{A.33}$$
$$S^o \;=\; T^o \cdot T^e : \; g^o \rightarrow f^o \tag{A.34}$$

A Stationary Rational Expectations Equilibrium in the periodic disturbances case is a pair of functions g^o, g^e satisfying $f^o = g^o$ and $f^e = g^e$. That is, $f^o = S^o f^o$ and $f^e = S^e f^e$.

The functions g^o, g^e are chosen to belong to the space $\mathbf{G}^*$ (a restriction of the space $\mathbf{G}$) characterized by:

Definition. $\mathbf{G}^*$ is the space of functions $g(x)$ such that:

(i) The domain of $g(x)$ is $X = \{x \,|\, x \in \mathbf{R},\ x \geq w_m\}$.

(ii) $g(x)$ is continuous, non-increasing and non-negative.

(iii) $g(w_m) = P(w_m)$.

A minor modification of the metric is also necessary:

Definition. The metric $d^*(g_i, g_j),\ g_i, g_j \in \mathbf{G}$ is defined by

$$d^*(g_i, g_j) \;=\; \sup_{x \in X} |g_i(x) - g_j(x)|.$$

The properties of $\mathbf{G}^*$ (in particular the boundedness of every $g \in \mathbf{G}^*$) guarantee that the metric d^* is well-defined.

The subsequent presentation is simplified by introducing functions $G^h,\ h = e, o$ each with domain $Y = \{(p, x) \,|\, P(x) \leq p < p_1,\ x \in X\}$, defined by:

$$G^h(q, x) = \theta \int_{W^h} g^h[w' + (1 - \delta)(x - D(q))] Q^h(dw'). \tag{A.35}$$

Lemma 5: S^e (resp. S^o) maps $\mathbf{G}^*$ into itself.

Proof. The proof is in four parts:

(4.a) For any $g^e \in \mathbf{G}^*$, it will be shown that $f^o = T^o g^e \in \mathbf{G}^*$.

(4.b) For any $g^o \in \mathbf{G}^*$, an identical argument to (4.a) shows that $f^e = T^e g^o \in \mathbf{G}^*$.

(4.c) In order to demonstrate that $S^e \equiv T^e \cdot T^o$ maps $\mathbf{G}^*$ into itself, set f^o from part (4.a) of the proof equal to g^o in part (4.b).

(4.d) In order to demonstrate that $S^o \equiv T^o \cdot T^e$ maps $\mathbf{G}^*$ into itself, set f^e from part (4.b) of the proof equal to g^e in part (4.a).

A proof of (4.a) is as follows:

For any given $x \in X$, $f^o(x)$ solves

$$\max_q [G^e(q,\, x) \,-\, q,\ P(x) \,-\, q] = 0. \tag{A.36}$$

The properties of the solution to (A.36) follow from the properties of $G^e(q,\, x)$. Since g^e is continuous, non-negative and non-increasing, it follows that G^e is continuous, non-negative and non-increasing in both q and x. Moreover, $G^e \,-\, q$ is strictly decreasing in q.

Note from (A.36) that its solution must satisfy $q \,\geq\, P(x)$. Suppose that $q \,>\, P(x)$; then, since g^e is non-increasing with p_1 as the upper-bound of its domain,

$$G^e(q,\, x) \,\leq\, G^e(P(x),\, x) \,=\, \theta \int_{W^e} g^e(w')Q^e(dw') \,\leq\, \theta p_1 \,<\, p_1$$

so that $G^e(q,\, x) \,-\, q$ tends to a negative number as $q \,\rightarrow\, p_1$.

At $q \,=\, P(x)$, if $G^e(q,\, x) \,-\, q \,=\, \theta \int_{W^e} g^e(w')Q^e(dw') \,-\, P(x) \leq\, 0$, it follows from the monotonicity of G^e that $G^e(q,\, x) \,-\, q$ is negative for all $q \,>\, P(x)$ and the solution to (A.36) must be $f^o(x) \,=\, P(x)$.

Alternatively, if at $q \,=\, P(x)$, $G^e(q,\, x) \,-\, q \,=\, \theta \int_{W^e} g^e(w')Q^e(dw') - P(x) \,>\, 0$, it follows, again from the monotonicity of G^e, that $G^e(q,\, x) \,-\, q$ has a unique zero root, $q \,=\, f^o(x)$ such that $G^e(f^o(x),\, x) \,-\, f^o(x) \equiv\, 0$.

Thus the solution to (A.36) provides a function $f^o(x)$ which is unique, continuous in x, non-increasing and non-negative for $x \,\in\, X$.

It remains to show that, if $g^e(w_m) \,=\, P(w_m)$, then $f^o(w_m) \,=\, P(w_m)$. Let $x \,=\, w_m$, so that $f^o(w_m)$ solves

$$\max_q [G^e(q,\, w_m) \,-\, q,\ P(w_m) \,-\, q] = 0. \tag{A.37}$$

From the reasoning above, $f^o(w_m) \,=\, P(w_m)$ requires that $G^e(P(w_m),\, w_m) - P(w_m) \,\leq\, 0$. That this must be so follows from the definition of G^e, the assumption that g^e is non-increasing, and assumption 2:

$$\begin{aligned} G^e(P(w_m),\, w_m) \,-\, P(w_m) \;&=\; \theta \int_{W^e} g^e(w')Q(dw') \,-\, P(w_m) \\ &\leq\; \theta g^e(w_m) \,-\, P(w_m) \\ &=\; (\theta \,-\, 1)P(w_m) \,<\, 0 \end{aligned} \tag{A.38}$$

which establishes that $f^o \,=\, T^o g^e \,\in\, \mathbf{G}^*$. $\|$

Lemma 6: S^e and S^o are contraction mappings.

Proof.

(5.a): S^e is a contraction mapping.

Choose any pair $g_0,\, g_2 \in \mathbf{G}^*$ such that

$$g_2(x) \leq g_0(x) \leq g_2(x) + a, \quad \forall x \in X \tag{A.39}$$

where a is a positive constant. It can now be shown that the mappings T^o and T^e preserve the inequalities.

Note, first, that $G_2 \leq G_0$ from the definition of G (where the subscripts correspond to different g functions but evaluated at the same x and q).

Fix $x \in X$ and let q_0 denote the (unique) root of $G_0(q,\, x) - q = 0$, if it exists. Since $G_2(q,\, x) \leq G_0(q,\, x)$, then $G_2(q_0,\, x) - q_0 \leq 0$. Hence the root, q_2, of $G_2(q,\, x) - q = 0$ must satisfy $P(x) \leq q_2 \leq q_0$, thus establishing that $T^o g_2 \leq T^o g_0$. Trivially, if either or both of the roots does not exist, the solution will be $q_0 = P(x)$ or $q_2 = P(x)$, so that it is just possible that $T^o g_2 = T^o g_0$.

Now consider $T^o(g_2 + a)$,

$$\begin{aligned} T^o(g_2 + a) &= \max\left[\theta \int_{W^e} g_2[w' + (1-\delta)(x - D(g_3(x)))]Q^e(dw') + \theta a, P(x)\right] \\ &\leq \theta a + \max\left[\theta \int_{W^e} g_2[w' + (1-\delta)(x - D(g_3(x)))]Q^e(dw'), P(x)\right] \\ &= \theta a + T^o g_2 \end{aligned} \tag{A.40}$$

Thus,

$$T^o g_2(x) \leq T^o g_0(x) \leq T^o g_2(x) + \theta a \tag{A.41}$$

which may be written,

$$g_3(x) \leq g_1(x) \leq g_3(x) + \theta a. \tag{A.42}$$

Note that, by construction, $G^o_3 \leq G^o_1$.

Fix $x \in X$ and let q_1 denote the (unique) root of $G_1(q,\, x) - q = 0$ if it exists. Since $G_3(q,\, x) \leq G_1$, then $G_3(q_1,\, x) - q_1 \leq 0$. Hence the root, q_3, of $G_3(q,\, x) - q = 0$ must satisfy $P(x) \leq q_3 \leq q_1$, thus establishing that $T^e g_3 \leq T^e g_1$. Trivially, if either or both of the roots does not exist, the solution will be $q_1 = P(x)$ or $q_3 = P(x)$, so that it is just possible that $T^o g_3 = T^o g_1$.

Consider $T^e(g_3 + \theta a)$,

$$\begin{aligned} T^e(g_3 + \theta a) &= \max\left[\theta \int_{W^o} g_3[w' + (1-\delta)(x - D(g_4(x)))]Q^o(dw') + \theta^2 a, P(x)\right] \\ &\leq \theta^2 a + \max\left[\theta \int_{W^o} g_3[w' + (1-\delta)(x - D(g_4(x)))]Q^o(dw'),\, P(x)\right] \\ &= \theta^2 a + T^e g_3 \end{aligned} \tag{A.43}$$

Thus,

$$T^e g_3(x) \leq T^e g_1(x) \leq T^e g_3(x) + \theta^2 a \tag{A.44}$$

and

$$S^e g_2(x) \leq S^e g_0(x) \leq S^e g_2(x) + \theta^2 a. \tag{A.45}$$

Now set $a = d^*(g_0,\, g_2)$ using the metric defined above. Then,

$$\begin{aligned} S^e g_0 - S^e g_2 &\leq \theta^2 a \\ |S^e g_0 - S^e g_2| &\leq \theta^2 a \quad \text{for every } x \in X \\ d^*(S^e g_0,\, S^e g_2) &\leq \theta^2 a \;=\; \theta^2 d^*(g_0,\, g_2) \end{aligned} \tag{A.46}$$

establishing that S^e is a contraction mapping.

(5.b) S^o is a contraction mapping. The proof is identical to that for (5.a). ||

Lemma 7: The space $\mathbf{G}^*$ is complete in the metric d^*.

Proof. This is simply a special case of the argument for lemma 3. ||

Theorem 4. *Under assumptions 1, 2, 3c there exist a unique pair of price functions* $f^e : X \to \mathbf{R}$, $f^o : X \to \mathbf{R}$, *such that* $f^e(\cdot)$, $f^o(\cdot)$ *are continuous, non-negative and non-increasing and satisfy:*

$$f^e(x) = \max\left[\theta \int_{W^o} f^o[w' + (1-\delta)(x - D(f^e(x)))]Q^o(dw'),\, P(x)\right] \tag{A.47}$$

$$f^o(x) = \max\left[\theta \int_{W^e} f^e[w' + (1-\delta)(x - D(f^o(x)))]Q^e(dw'),\, P(x)\right]. \tag{A.48}$$

Proof. The proof demonstrates that S^e (respectively, S^o) has a unique fixed point.

Choose any $g_0,\, g_1 \in \mathbf{G}^*$ and consider the sequences:

$$\begin{array}{llllll} g_0, & g_2 = S^e g_0, & g_4 = S^e g_2, & \ldots, & g_t = S^e g_{t-2}, & \ldots \quad \text{for } t \text{ even;} \\ g_1, & g_3 = S^o g_1, & g_5 = S^o g_3, & \ldots, & g_t = S^o g_{t-2}, & \ldots \quad \text{for } t \text{ odd.} \end{array}$$

For even t the definition of d^* implies that $d^*(g_{t-2},\, g_t) \leq \theta^{t-2} d^*(g_0,\, g_2)$. Hence, from assumption 2, the sequence converges uniformly. Denote the limit of the sequence by f^e. From lemma 7, $f^e \in \mathbf{G}^*$.

To show that the limit is an equilibrium, that is, $S^e f^e = f^e$, consider

$$\begin{aligned} d^*(S^e f^e,\, f^e) &\leq d^*(S^e f^e,\, (S^e)^n g_0) + d^*((S^e)^n g_0,\, f^e) \\ &\leq \theta^2 d^*(f^e,\, (S^e)^{n-1} g_0) + d^*((S^e)^n g_0,\, f^e) \quad \forall\, n,\ g_0 \in \mathbf{G}^* \end{aligned} \tag{A.49}$$

Both terms on the right hand side of the inequality tend to zero as $n \to \infty$, so that $d^*(S^e f^e,\, f^e) = 0$ and f^e is an equilibrium.

To show that the equilibrium is unique, suppose that $\tilde{f}^e \in \mathbf{G}^*$ is another solution $S^e \tilde{f}^e = \tilde{f}^e$. Then,

$$0 < \alpha = d^*(\tilde{f}^e,\, f^e) = d^*(S^e \tilde{f}^e,\, S^e f^e) \leq \theta^2 d^*(\tilde{f},\, f^e) = \theta^2 \alpha$$

which is a contradiction. Hence f^e is unique.

For odd t, $d^*(g_{t-2},\, g_t) \leq \theta^{t-2} d^*(g_1,\, g_3)$ and an identical argument demonstrates that the limit of the sequence of g_t functions, say f^o, is the unique equilibrium for odd numbered periods. ‖

The multi-period case.

The extension from two to $n < +\infty$ periods is almost entirely notational. Each period is identified by a superscript $k = 1, 2, \ldots, n$ so that each harvest, w^k, is assumed to be a member of a set,

$$W^k = \{w \in \mathbf{R} \,|\, -\infty < \underline{w}^k \leq w \leq \overline{w}^k < +\infty\} \tag{A.50}$$

and $(W^k,\, \mathcal{W}^k)$ is a measurable space, where $\mathcal{W}^k$ is the Borel algebra of sets for W^k.

Assumption 3d (periodic disturbances).

The disturbances $w^k,\ k = 1, 2, \ldots, n$ are independently distributed. That is, the functions $Q^k : \mathcal{W}^k \rightarrow [0,\, 1]$ are probability measures on $\mathcal{W}^k$.

The lower bound, w_*, in assumption 1 now becomes $w_* = \min[\underline{w}^1,\, \underline{w}^2,\, \ldots,\, \underline{w}^n]$, and assumption 2 remains unchanged.

The mappings analogous to (A.32) are defined as follows:

$$f^1(x) = \max\left[\theta \int_{W^2} g^2[w' + (1-\delta)(x - D(f^1(x)))]Q^2(dw'),\, P(x)\right], \qquad T^1 : g^2 \rightarrow f^1 \tag{A.51}$$

$$f^2(x) = \max\left[\theta \int_{W^3} g^3[w' + (1-\delta)(x - D(f^2(x)))]Q^3(dw'),\, P(x)\right], \qquad T^2 : g^3 \rightarrow f^2 \tag{A.52}$$

$$\vdots$$

$$f^n(x) = \max\left[\theta \int_{W^1} g^1[w' + (1-\delta)(x - D(f^m(x)))]Q^1(dw'),\, P(x)\right], \qquad T^n : g^1 \rightarrow f^n \tag{A.53}$$

In order to link the corresponding periods it is necessary to define a composite mapping S^i for each i in the obvious way:

$$S^i = T^i \cdot T^{i+1} \cdots T^{n-1} \cdot T^n \cdot T^1 \cdots T^{i-2} \cdot T^{i-1} : g^i \rightarrow f^i.$$

A Stationary Rational Expectations Equilibrium in the n-period case is a set of n

functions $\{g^1, g^2, \ldots, g^n\}$ satisfying $f^i = g^i$, $i = 1, 2, \ldots, n$. That is, $f^i = S^i f^i$ for each period.

Theorem 4′. *Under assumptions 1, 2, 3d a unique SREE in the n-period case exists.*

Proof. The argument simply replicates the steps in the two-period case for the appropriate number of periods using the composite S^i mappings. ‖