Locally Exact Discrete Time Representations of Non-Linear Continuous Time Models with an Application to the Estimation of a DSGE Model: Supplementary Material

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Here we provide proofs of Propositions 1 and 2 as well as a derivation of (40) and (41).

Proof of Proposition 1. We begin by examining the difference

$$\begin{aligned}
\xi^a_{th+h} - \xi_{th+h} &= \left(y(th+h) - y(th) - \int_{th}^{th+h} f\big(y(r)\big)dr \right) \\
&\quad - \big(y(th+h) - \Theta_{th} y(th) - \Upsilon_{1,th+h} C_{th} - \Upsilon_{2,th+h}\Delta_{th} \big) \\
&= \left(\Theta_{th} - I_n\right) y(th) + \Upsilon_{1,th+h} C_{th} + \Upsilon_{2,th+h}\Delta_{th} - \int_0^h f\big(y(th+h-s)\big)ds
\end{aligned}$$

where a change of variable to $s = th + h - r$ has resulted in the final expression for the integral involving $f(\cdot)$. Define the following matrices:

$$P_{1,th} = \int_0^h s e^{A_{th}s} ds, \quad P_{2,th} = \int_0^h e^{A_{th}s} ds.$$

Then, by the same change of variable as above, we obtain

$$\begin{aligned}
\Upsilon_{1,th+h} &= \int_{th}^{th+h} e^{A_{th}(th+h-r)} r dr = \int_0^h e^{A_{th}s}(th+h-s)ds = (th+h)P_{2,th} - P_{1,th}, \\
\Upsilon_{2,th+h} &= \int_{th}^{th+h} e^{A_{th}(th+h-r)} dr = \int_0^h e^{A_{th}s} ds = P_{2,th}.
\end{aligned}$$

Combining the above and using the definition of Δ_{th} we then find that

$$\begin{aligned}
&\Upsilon_{1,th+h} C_{th} + \Upsilon_{2,th+h}\Delta_{th} \\
&\quad = \big((th+h)P_{2,th} - P_{1,th}\big)C_{th} + P_{2,th}\Big(f\big(y(th)\big) - A_{th} y(th) - C_{th} th \Big) \\
&\quad = \left(hP_{2,th} - P_{1,th}\right) C_{th} + P_{2,th}\Big(f\big(y(th)\big) - A_{th} y(th) \Big).
\end{aligned}$$

Hence, recalling that $\Theta_{th} = e^{A_{th}h}$, we find that

$$\begin{aligned}
\xi^a_{th+h} - \xi_{th+h} &= \left(e^{A_{th}h} - I_n\right) y(th) + \left(hP_{2,th} - P_{1,th}\right) C_{th} \\
&\quad + P_{2,th}\Big(f\big(y(th)\big) - A_{th}y(th)\Big) - \int_0^h f\big(y(th+h-s)\big)ds \\
&= \left(e^{A_{th}h} - I_n - P_{2,th}A_{th}\right) y(th) + \left(hP_{2,th} - P_{1,th}\right) C_{th} \\
&\quad + P_{2,th} f\big(y(th)\big) - \int_0^h f\big(y(th+h-s)\big)ds. \qquad \text{(S1)}
\end{aligned}$$

In what follows we shall make use of the following expansions:

$$\begin{aligned}
P_{1,th} = \int_0^h s e^{A_{th}s} ds &= \int_0^h s \left(\sum_{j=0}^{\infty} \frac{(A_{th}s)^j}{j!}\right) ds \\
&= \sum_{j=0}^{\infty} \left(\int_0^h s^{j+1} ds\right) \frac{A_{th}^j}{j!} \\
&= \sum_{j=0}^{\infty} \frac{h^{j+2} A_{th}^j}{(j+2)j!} \\
&= \sum_{j=2}^{\infty} \frac{h^j A_{th}^{j-2}}{j(j-2)!}, \\
P_{2,th} = \int_0^h e^{A_{th}s} ds &= \int_0^h \left(\sum_{j=0}^{\infty} \frac{(A_{th}s)^j}{j!}\right) ds \\
&= \sum_{j=0}^{\infty} \left(\int_0^h s^j ds\right) \frac{A_{th}^j}{j!} \\
&= \sum_{j=0}^{\infty} \frac{h^{j+1} A_{th}^j}{(j+1)j!} \\
&= \sum_{j=1}^{\infty} \frac{h^j A_{th}^{j-1}}{j!}.
\end{aligned}$$

Using the second result we find that the first term in (S1) is zero because[1]

$$e^{A_{th}h} - I_n - P_{2,th}A_{th} = I_n + \sum_{j=1}^{\infty} \frac{h^j A_{th}^j}{j!} - I_n - \left(\sum_{j=1}^{\infty} \frac{h^j A_{th}^{j-1}}{j!}\right) A_{th} = 0$$

[1]This is a generalisation of the result that, if A is nonsingular, then $\int_0^h e^{As} ds = A^{-1}(e^{hA} - I_n)$, which can be written in the form $e^{hA} - I_n - \int_0^h e^{As} ds A$ using the commutability of e^A and A.

and so

$$\begin{aligned}
E_{th}\left\|\xi^a_{th+h}-\xi_{th+h}\right\| &\leq \left\|(hP_{2,th}-P_{1,th})\,C_{th}\right\|+\left\|P_{2,th}f\big(y(th)\big)\right\| \\
&\quad +E_{th}\left\|\int_0^h f\big(y(th+h-s)\big)ds\right\| \\
&\leq \left\|hP_{2,th}-P_{1,th}\right\|\left\|C_{th}\right\|+\left\|P_{2,th}\right\|\left\|f\big(y(th)\big)\right\| \\
&\quad +\int_0^h E_{th}\left\|f\big(y(th+h-s)\big)\right\|ds. \qquad \text{(S2)}
\end{aligned}$$

From the series expansions of $P_{1,th}$ and $P_{2,th}$ it is clear that $P_{2,th}=O(h)$ while

$$\begin{aligned}
hP_{2,th}-P_{1,th} &= h\sum_{j=1}^{\infty}\frac{h^jA_{th}^{j-1}}{j!}-\sum_{j=2}^{\infty}\frac{h^jA_{th}^{j-2}}{j(j-2)!} \\
&= h\left(hI_n+\frac{h^2A_{th}}{2}+\ldots\right)-\left(\frac{h^2}{2}I_n+\frac{h^3A_{th}}{3}+\ldots\right)=\frac{h^2}{2}I_n+O(h^3)
\end{aligned}$$

and so this matrix is $O(h^2)$. It then follows that

$$E_{th}\left\|\xi^a_{th+h}-\xi_{th+h}\right\|\leq O(h)+O(h^2)+O(h)=O(h)$$

under the conditions stated in the Proposition. □

Proof of Proposition 2. The proof of Proposition 2 follows in a similar way to that of Proposition 1. Using Theorem 2 and (19) we find that

$$\begin{aligned}
\bar{\xi}^a_{th+h}-\bar{\xi}_{th+h} &= \big(\bar{y}_{th+h}-\bar{y}_{th}-\bar{F}_{th+h}\big)-\big(\bar{y}_{th+h}-\bar{\Theta}_{th}\bar{y}_{th}-\bar{\Upsilon}_{1,th+h}\bar{C}_{th}-\bar{\Upsilon}_{2,th+h}\bar{\Delta}_{th}\big) \\
&= \big(\bar{\Theta}_{th}-I_n\big)\,\bar{y}_{th}+\bar{\Upsilon}_{1,th+h}\bar{C}_{th}+\bar{\Upsilon}_{2,th+h}\bar{\Delta}_{th}-\bar{F}_{th+h}.
\end{aligned}$$

Using the definitions of $\bar{K}_{1,th}(r)$ and $\bar{K}_{2,th}(r)$ in Theorem 2 we obtain

$$\bar{\Upsilon}_{2,th+h}=\int_{th}^{th+h}\int_0^{th+h-r}e^{\bar{A}_{th}s}dsdr+\int_{th-h}^{th}\int_{th-r}^{h}e^{\bar{A}_{th}s}dsdr.$$

Using the change of variable $w=th+h-r$ in the first integral and $w=th-r$ in the second it follows that

$$\begin{aligned}
\bar{\Upsilon}_{2,th+h} &= \int_0^h\int_0^w e^{\bar{A}_{th}s}dsdw+\int_0^h\int_w^h e^{\bar{A}_{th}s}dsdw \\
&= \int_0^h\int_0^h e^{\bar{A}_{th}s}dsdw \\
&= h\bar{P}_{2,th}
\end{aligned}$$

where

$$\bar{P}_{2,th}=\int_0^h e^{\bar{A}_{th}s}ds.$$

Applying similar reasoning to $\bar{\Upsilon}_{1,th+h}$ yields

$$\begin{aligned}
\bar{\Upsilon}_{1,th+h} &= \int_0^h \int_0^w e^{\bar{A}_{th}s} ds(th+h-w)dw + \int_0^h \int_w^h e^{\bar{A}_{th}s} ds(th-w)dw \\
&= (th+h) \int_0^h \int_0^w e^{\bar{A}_{th}s} dsdw - \int_0^h \int_0^w e^{\bar{A}_{th}s} dswdw \\
&\quad + th \int_0^h \int_w^h e^{\bar{A}_{th}s} dsdw - \int_0^h \int_w^h e^{\bar{A}_{th}s} dswdw \\
&= th\bar{\Upsilon}_{2,th+h} + h \int_0^h \int_0^w e^{\bar{A}_{th}s} dsdw - \int_0^h \int_0^h e^{\bar{A}_{th}s} dswdw.
\end{aligned}$$

Combining these expressions for $\bar{\Upsilon}_{1,th+h}$ and $\bar{\Upsilon}_{2,th+h}$ with $\bar{C}_{th}$ and the definition of $\bar{\Delta}_{th}$ in Theorem 2 gives

$$\begin{aligned}
\bar{\Upsilon}_{1,th+h}\bar{C}_{th} + \bar{\Upsilon}_{2,th+h}\bar{\Delta}_{th} &= \left(th\bar{\Upsilon}_{2,th+h} + h \int_0^h \int_0^w e^{\bar{A}_{th}s} dsdw - \int_0^h \int_0^h e^{\bar{A}_{th}s} dswdw \right) \bar{C}_{th} \\
&\quad + \bar{\Upsilon}_{2,th+h} \left(f(\bar{y}_{th}) - \bar{A}_{th}\bar{y}_{th} - \bar{C}_{th}th \right) \\
&= \left(h \int_0^h \int_0^w e^{\bar{A}_{th}s} dsdw - \int_0^h \int_0^h e^{\bar{A}_{th}s} dswdw \right) \bar{C}_{th} \\
&\quad + h\bar{P}_{2,th} \left(f(\bar{y}_{th}) - \bar{A}_{th}\bar{y}_{th} \right).
\end{aligned}$$

Turning to $\bar{F}_{th+h}$ it is possible to simplify the double integral as follows:

$$\begin{aligned}
\bar{F}_{th+h} &= \int_{th-h}^{th} \int_{th}^{r+h} f\big(y(r)\big) dsdr + \int_{th}^{th+h} \int_r^{th+h} f\big(y(r)\big) dsdr \\
&= \int_{th-h}^{th} (r+h-th) f\big(y(r)\big) dr + \int_{th}^{th+h} (th+h-r) f\big(y(r)\big) dr \\
&= \int_0^h (h-u) f\big(y(th-u)\big) du + \int_0^h u f\big(y(th+h-u)\big) du
\end{aligned}$$

where the final integrals are obtained by the changes of variable to $u = th - r$ and $u = th + h - r$, respectively. Using these results we obtain

$$\begin{aligned}
\bar{\xi}^a_{th+h} - \bar{\xi}_{th+h} &= \left(\bar{\Theta}_{th} - I_n - h\bar{P}_{2,th}\bar{A}_{th} \right) \bar{y}_{th} + h\bar{P}_{2,th} f(\bar{y}_{th}) \\
&\quad + \int_0^h \left(h \int_0^w e^{\bar{A}_{th}s} ds - w \int_0^h e^{\bar{A}_{th}s} ds \right) dw \, \bar{C}_{th} \\
&\quad + \int_0^h (h-u) f\big(y(th-u)\big) du + \int_0^h u f\big(y(th+h-u)\big) du
\end{aligned}$$

and so the quantity of interest satisfies

$$\begin{aligned} E_{th}\left\|\bar{\xi}^a_{th+h}-\bar{\xi}_{th+h}\right\| &\le \left\|\bar{\Theta}_{th}-I_n-h\bar{P}_{2,th}\bar{A}_{th}\right\|\left\|\bar{y}_{th}\right\|+h\left\|\bar{P}_{2,th}\right\|\left\|f(\bar{y}_{th})\right\| \\ &\quad +\int_0^h\left\|h\int_0^w e^{\bar{A}_{th}s}ds-w\int_0^h e^{\bar{A}_{th}s}ds\right\|dw\left\|\bar{C}_{th}\right\| \\ &\quad +h\int_0^h\left\|f\big(y(th-u)\big)\right\|du+h\int_0^h\left\|f\big(y(th+h-u)\big)\right\|du. \end{aligned}$$

The first term involves the matrix

$$\begin{aligned} \bar{\Theta}_{th}-I_n-h\bar{P}_{2,th}\bar{A}_{th} &= e^{\bar{A}_{th}h}-I_n-h\sum_{j=1}^{\infty}\frac{h^j\bar{A}_{th}^{j-1}}{j!}\bar{A}_{th} \\ &= \sum_{j=1}^{\infty}\frac{h^j\bar{A}_{th}^{j}}{j!}-h\sum_{j=1}^{\infty}\frac{h^j\bar{A}_{th}^{j}}{j!} \\ &= (1-h)\sum_{j=1}^{\infty}\frac{h^j\bar{A}_{th}^{j}}{j!}=O(h) \end{aligned}$$

while the second term involves

$$h\bar{P}_{2,th}=h\sum_{j=1}^{\infty}\frac{h^j\bar{A}_{th}^{j-1}}{j!}=O(h^2).$$

In the third term, consider

$$\begin{aligned} \left\|h\int_0^w e^{\bar{A}_{th}s}ds-w\int_0^h e^{\bar{A}_{th}s}ds\right\| &\le h\left\|\int_0^w e^{\bar{A}_{th}s}ds\right\|+w\left\|\int_0^h e^{\bar{A}_{th}s}ds\right\| \\ &\le h\int_0^w\left\|e^{\bar{A}_{th}s}\right\|ds+w\int_0^h\left\|e^{\bar{A}_{th}s}\right\|ds \\ &\le (h+w)\int_0^h\left\|e^{\bar{A}_{th}s}\right\|ds. \end{aligned}$$

Then

$$\begin{aligned} \int_0^h\left\|h\int_0^w e^{\bar{A}_{th}s}ds-w\int_0^h e^{\bar{A}_{th}s}ds\right\|dw &\le \int_0^h(h+w)\int_0^h\left\|e^{\bar{A}_{th}s}\right\|dsdw \\ &= h^2\int_0^h\left\|e^{\bar{A}_{th}s}\right\|ds+\frac{h^2}{2}\int_0^h\left\|e^{\bar{A}_{th}s}\right\|ds \\ &= \frac{3h^2}{2}\int_0^h\left\|e^{\bar{A}_{th}s}\right\|ds=O(h^3). \end{aligned}$$

Hence, given the stated assumptions and noting that $\|\bar{y}_{th}\|=O(h)$, we have

$$E_{th}\left\|\bar{\xi}^a_{th+h}-\bar{\xi}_{th+h}\right\|\le O(h^2)+O(h^2)+O(h^3)+O(h^2)+O(h^2)=O(h^2)$$

as claimed. □

Derivation of (40) **and** (41)**.** We provide details of the derivation of (40) and (41); details of the derivation of the remaining laws of motion in (32), (42) and (45) are provided in the text. Subscripted variables will denote partial derivatives and the dependence on time (t) will be suppressed in order to aid readability so that, for example, $V_k = \partial V(k,\varpi)/\partial k$ and the variables are c, h, k, ϖ, z and r. Let $c(k,\varpi)$ and $h(k,\varpi)$ denote the optimal values of c and h obtained from the optimisation problem in (37). The maximised value of the Bellman equation is then

$$\rho V(k,\varpi) = \Big\{ u\big(c(k,\varpi), h(k,\varpi)\big) + \Big(\varpi k^{\alpha} h(k,\varpi)^{1-\alpha} - c(k,\varpi) - \big(\delta + \mu_N - \sigma_N^2\big)k\Big)V_k + \frac{1}{2}\big(\sigma_K^2 + \sigma_N^2\big)k^2 V_{kk} + \mu_{\varpi}\varpi V_{\varpi} + \frac{1}{2}\sigma_{\varpi}^2 \varpi^2 V_{\varpi\varpi} \Big\}.$$

Differentiating with respect to k and suppressing dependence on k and ϖ where appropriate for notational convenience we obtain, after rearrangement of terms,

$$\begin{aligned} &\big(\rho - \alpha\varpi k^{\alpha-1} h^{1-\alpha} + (\delta + \mu_N - \sigma_N^2)\big)V_k \\ &\quad = \big(\varpi k^{\alpha} h^{1-\alpha} - c - (\delta + \mu_N - \sigma_N^2)k + (\sigma_K^2 + \sigma_N^2)k\big)V_{kk} \\ &\qquad + \frac{1}{2}(\sigma_K^2 + \sigma_N^2)k^2 V_{kkk} + \mu_{\varpi}\varpi V_{\varpi k} + \frac{1}{2}\sigma_{\varpi}^2\varpi^2 V_{\varpi\varpi k}, \end{aligned} \tag{S3}$$

where we have also utilised the first-order conditions following (37) to eliminate some terms involving derivatives of the utility function. Application of Ito's formula to V_k yields

$$dV_k = V_{kk}dk + V_{k\varpi}d\varpi + \frac{1}{2}V_{kkk}(dk)^2 + \frac{1}{2}V_{k\varpi\varpi}(d\varpi)^2. \tag{S4}$$

The constraint (36) determines dk while (32) implies that

$$d\varpi = \mu_{\varpi}\varpi dt + \sigma_{\varpi}\varpi dw_{\varpi},$$

while we also have

$$(dk)^2 = (\sigma_K^2 + \sigma_N^2)k^2 dt, \quad (d\varpi)^2 = \sigma_{\varpi}^2\varpi^2 dt.$$

Using these expressions in (S4) we obtain

$$\begin{aligned} dV_k &= \big(\varpi k^{\alpha}h^{1-\alpha} - c - (\delta + \mu_N - \sigma_N^2)k\big)V_{kk}dt + \sigma_K k V_{kk} dw_K - \sigma_N k V_{kk} dw_N \\ &\quad + \mu_{\varpi}\varpi V_{k\varpi}dt + \sigma_{\varpi}\varpi V_{k\varpi}dw_{\varpi} + \frac{1}{2}(\sigma_K^2 + \sigma_N^2)k^2 V_{kkk}dt + \frac{1}{2}\sigma_{\varpi}^2\varpi^2 V_{k\varpi\varpi}dt. \end{aligned} \tag{S5}$$

From (S3) we obtain

$$\begin{aligned} &\big(\varpi k^{\alpha}h^{1-\alpha} - c - (\delta + \mu_N - \sigma_N^2)k\big)V_{kk} \\ &\quad = \big(\rho - \alpha\varpi k^{\alpha-1}h^{1-\alpha} + (\delta + \mu_N - \sigma_N^2)\big)V_k - (\sigma_K^2 + \sigma_N^2)kV_{kk} \\ &\qquad - \frac{1}{2}(\sigma_K^2 + \sigma_N^2)k^2 V_{kkk} - \mu_{\varpi}\varpi V_{\varpi k} - \frac{1}{2}\sigma_{\varpi}^2\varpi^2 V_{\varpi\varpi k}, \end{aligned} \tag{S6}$$

the right-hand side of which can be substituted for the first term on the right-hand side of

(S5) to give

$$
\begin{aligned}
dV_k &= \big(\rho - \alpha\varpi k)^{\alpha-1}h^{1-\alpha} + (\delta + \mu_N - \sigma_N^2)\big)V_k dt - (\sigma_K^2 + \sigma_N^2)kV_{kk}dt \\
&\quad -\frac{1}{2}(\sigma_K^2 + \sigma_N^2)k^2 V_{kkk}dt - \mu_\varpi \varpi V_{\varpi k}dt - \frac{1}{2}\sigma_\varpi^2 \varpi^2 V_{\varpi\varpi k}dt \\
&\quad +\sigma_K k V_{kk} dw_K - \sigma_N k V_{kk} dw_N + \mu_\varpi \varpi V_{k\varpi}dt + \sigma_\varpi \varpi V_{k\varpi} dw_\varpi \\
&\quad +\frac{1}{2}(\sigma_K^2 + \sigma_N^2)k^2 V_{kkk}dt + \frac{1}{2}\sigma_\varpi^2 \varpi^2 V_{k\varpi\varpi}dt
\end{aligned} \tag{S7}
$$

which simplifies to

$$
\begin{aligned}
dV_k &= \big(\rho - \alpha\varpi k^{\alpha-1}h^{1-\alpha} + (\delta + \mu_N - \sigma_N^2)\big)V_k dt - (\sigma_K^2 + \sigma_N^2)kV_{kk}dt \\
&\quad +\sigma_K k V_{kk} dw_K - \sigma_N k(t) V_{kk} dw_N + \sigma_\varpi \varpi V_{k\varpi} dw_\varpi
\end{aligned} \tag{S8}
$$

upon cancellation of terms.

We now translate the above equation for V_k into one for u_c from which we can derive (40). Differentiating the first-order condition (38) with respect to both ϖ and k we obtain

$$
V_{kk} = u_{cc} c_\varpi, \quad V_{kk} = u_{cc} c_k,
$$

while (38) itself implies that $dV_k = du_c$. Substituting these quantities into (S8) results in

$$
\begin{aligned}
du_c &= \big(\rho - \alpha\varpi k^{\alpha-1}h^{1-\alpha} + (\delta + \mu_N - \sigma_N^2)\big)u_c dt - (\sigma_K^2 + \sigma_N^2)k u_{cc} c_k dt \\
&\quad +u_{cc} c_k \big(\sigma_K k dw_K - \sigma_N k dw_N\big) + \sigma_\varpi \varpi u_{cc} c_\varpi dw_\varpi.
\end{aligned} \tag{S9}
$$

Noting that $du_c/c = u_{cc} dc/c$ we divide (S9) through by cu_{cc} to obtain an equation for dc/c:

$$
\begin{aligned}
\frac{dc}{c} &= \big(\rho - \alpha\varpi k^{\alpha-1}h^{1-\alpha} + (\delta + \mu_N - \sigma_N^2)\big)\frac{u_c}{cu_{cc}}dt - (\sigma_K^2 + \sigma_N^2)\epsilon_{ck} dt \\
&\quad +\epsilon_{ck}\big(\sigma_K dw_K - \sigma_N dw_N\big) + \epsilon_{c\varpi}\sigma_\varpi dw_\varpi
\end{aligned} \tag{S10}
$$

where ϵ_{ck} and $\epsilon_{c\varpi}$ are the elasticities

$$
\epsilon_{ck} = c_k \frac{k}{c}, \quad \epsilon_{c\varpi} = c_\varpi \frac{\varpi}{c}.
$$

From the form of utility function in (34) it is straightforward to show that $u_c/u_{cc} = -c/\eta$ and so (S10) becomes

$$
\begin{aligned}
\frac{dc}{c} &= \frac{1}{\eta}\big(\alpha\varpi k^{\alpha-1}h^{1-\alpha} - (\delta + \mu_N - \sigma_N^2) - \rho\big)dt - (\sigma_K^2 + \sigma_N^2)\epsilon_{ck} dt \\
&\quad +\epsilon_{ck}\big(\sigma_K dw_K - \sigma_N dw_N\big) + \epsilon_{c\varpi}\sigma_\varpi dw_\varpi.
\end{aligned} \tag{S11}
$$

We now consider a candidate functional form for the value function:

$$
V(k, \varpi) = \frac{g}{1-\alpha\eta} k^{1-\alpha\eta} \varpi^{-\eta}.
$$

This function, along with the utility function in (34), enables (38) to be written

$$c^{-\eta}(1-h)^{(1-\eta)\psi} = gk^{-\alpha\theta}\varpi^{-\eta}$$

which can be solved for c to give

$$c = g^{-1/\eta}\varpi k^{\alpha}(1-h)^{(1-\eta)\psi/\eta}. \tag{S12}$$

From this expression it can be shown that $\epsilon_{ck} = \alpha$ and $\epsilon_{c\varpi} = 1$ and hence (S11) becomes

$$\begin{aligned}\frac{dc}{c} &= \frac{1}{\eta}\big(\alpha\varpi k^{\alpha-1}h^{1-\alpha} - (\delta + \mu_N - \sigma_N^2) - \rho - \alpha\eta(\sigma_K^2 + \sigma_N^2)\big)dt \\ &\quad + \alpha\big(\sigma_K dw_K - \sigma_N dw_N\big) + \sigma_{\varpi} dw_{\varpi}.\end{aligned} \tag{S13}$$

Noting that the first term in (S13) is the (per capita) marginal product of capital, which in turn is equal to the rental rate of capital, $r(t)$, enables (S13) to be written as (40). The steps involved in writing (40) in the final form (46) are outlined in the main text.

Turning to (41) we begin by taking the total differential of (39), with $w = z_h$, to obtain

$$dV_k = -\left(\frac{z_h du_h - u_h(z_{hh}dh + z_{hk}dk)}{z_h^2}\right).$$

Using (S8) to substitute for dV_k we can write

$$\begin{aligned}-\left(\frac{z_h du_h - u_h(z_{hh}dh + z_{hk}dk)}{z_h^2}\right) &= \big(\rho - \alpha\varpi k^{\alpha-1}h^{1-\alpha} + (\delta + \mu_N - \sigma_N^2)\big)V_k dt \\ &\quad -(\sigma_K^2 + \sigma_N^2)kV_{kk}dt + \sigma_K kV_{kk}dw_K - \sigma_N kV_{kk}dw_N(t) + \sigma_{\varpi}\varpi V_{k\varpi}dw_{\varpi};\end{aligned}$$

multiplying by z_h and noting from (39) that $z_h V_k = -u_h$ we obtain

$$\begin{aligned}-\left(\frac{z_h du_h - u_h(z_{hh}dh + z_{hk}dk)}{z_h}\right) &= -\big(\rho - \alpha\varpi k^{\alpha-1}h^{1-\alpha} + (\delta + \mu_N - \sigma_N^2)\big)u_h dt \\ &\quad -(\sigma_K^2 + \sigma_N^2)kz_h V_{kk}dt + \sigma_K kz_h V_{kk}dw_K - \sigma_N kz_h V_{kk}dw_N + \sigma_{\varpi}\varpi z_h V_{k\varpi}dw_{\varpi}.\end{aligned}$$

Noting that $r = \alpha\varpi k^{\alpha-1}h^{1-\alpha}$ while from (39) we obtain

$$z_h V_{kk} = -u_{hh}\frac{\partial h}{\partial k}, \quad z_h V_{k\varpi} = -u_{hh}\frac{\partial h}{\partial \varpi},$$

enables the equation to be written

$$\begin{aligned}-du_h + u_h\frac{(z_{hh}dh + z_{hk}dk)}{z_h} &= -\big(\rho - r + (\delta + \mu_N - \sigma_N^2)\big)u_h dt \\ &\quad +(\sigma_K^2 + \sigma_N^2)ku_{hh}\frac{\partial h}{\partial k}dt - \sigma_K ku_{hh}\frac{\partial h}{\partial k}dw_K + \sigma_N ku_{hh}\frac{\partial h}{\partial k}dw_N - \sigma_{\varpi}\varpi u_{hh}\frac{\partial h}{\partial \varpi}dw_{\varpi}.\end{aligned} \tag{S14}$$

Now, from (29), we obtain

$$z_h = (1-\alpha)\varpi k^{\alpha}h^{-\alpha}, \quad z_{hh} = -\alpha(1-\alpha)\varpi k^{\alpha}h^{-\alpha-1}, \quad z_{hk} = \alpha(1-\alpha)\varpi k^{\alpha-1}h^{-\alpha},$$

and hence

$$\frac{(z_{hh}dh + z_{hk}dk)}{z_h} = \alpha\frac{dk}{k} - \alpha\frac{dh}{h}.$$

Substituting this expression into (S14) and multiplying by -1 then yields

$$du_h = \alpha u_h\left(\frac{dk}{k} - \frac{dh}{h}\right) + \big(\rho - r + (\delta + \mu_N - \sigma_N^2)\big)u_h dt$$
$$-(\sigma_K^2 + \sigma_N^2)ku_{hh}\frac{\partial h}{\partial k}dt + \sigma_K k u_{hh}\frac{\partial h}{\partial k}dw_K - \sigma_N k u_{hh}\frac{\partial h}{\partial k}dw_N + \sigma_\varpi \varpi u_{hh}\frac{\partial h}{\partial \varpi}dw_\varpi. \text{(S15)}$$

We now substitute for dk/k using (42) and divide by u_h to obtain (after some simplification)

$$\frac{du_h}{u_h} = \left(\rho - \alpha\frac{c}{k} - (\alpha+1)(\delta + \mu_N - \sigma_N^2)\right)dt - \alpha\frac{dh}{h} + \alpha\big(\sigma_K dw_K - \sigma_N dw_N\big)$$
$$+\frac{u_{hh}}{u_h}\frac{\partial h}{\partial k}\left(\sigma_K k dw_K - \sigma_N k dw_N - (\sigma_K^2 + \sigma_N^2)kdt\right) + \frac{u_{hh}}{u_h}\frac{\partial h}{\partial \varpi}\sigma_\varpi \varpi dw_\varpi. \text{ (S16)}$$

Now, $du_h = u_{hh}dh$, so using this on the left-hand-side of (S16) and multiplying both sides by $u_h/(hu_{hh})$ we obtain

$$\frac{dh}{h} = \left(\rho - \alpha\frac{c}{k} - (\alpha+1)(\delta + \mu_N - \sigma_N^2)\right)\frac{u_h}{hu_{hh}}dt - \alpha\frac{u_h}{hu_{hh}}\frac{dh}{h}$$
$$+\alpha\frac{u_h}{hu_{hh}}\big(\sigma_K dw_K - \sigma_N dw_N\big) + \epsilon_{hk}\left(\sigma_K dw_K - \sigma_N dw_N - (\sigma_K^2 + \sigma_N^2)dt\right) + \epsilon_{h\varpi}\sigma_\varpi dw_\varpi \text{(S17)}$$

where $\epsilon_{hk} = (\partial h/\partial k)/(h/k)$ and $\epsilon_{h\varpi} = (\partial h/\partial \varpi)/(h/\varpi)$ denote the relevant elasticities. From the form of utility function in (34) it is straightforward to show that

$$\frac{u_h}{u_{hh}} = \frac{1-h}{1 + (\eta - 1)\psi};$$

substituting this into (S17) and simplifying we find that

$$\frac{dh}{h} = \left(\rho - \alpha\frac{c}{k} - (\alpha+1)(\delta + \mu_N - \sigma_N^2)\right)f(h)dt - \alpha f(h)\frac{dh}{h}$$
$$+\alpha f(h)\big(\sigma_K dw_K - \sigma_N dw_N\big)$$
$$+\epsilon_{hk}\left(\sigma_K dw_K - \sigma_N dw_N - (\sigma_K^2 + \sigma_N^2)dt\right) + \epsilon_{h\varpi}\sigma_\varpi dw_\varpi. \qquad \text{(S18)}$$

The next step is to eliminate ϵ_{hk} and $\epsilon_{h\varpi}$ from this equation. We make the assumption that there is a constant savings ratio, s, so that $c = (1-s)z$. Using the definition of z, the expression for c in (S12) implies that

$$c = \left(g^{-1/\eta}h^{\alpha-1}(1-h)^{(1-\eta)\psi/\eta}\right)z.$$

For this to be consistent with the assumed constant savings ratio requires that

$$g^{1/\eta} = (1-s)^{-1}h^{\alpha-1}(1-h)^{(1-\eta)\psi/\eta}.$$

We then find that $c = (1-s)\varpi k^\alpha h^{1-\alpha}$, the logarithm of which can be solved for $\log h$ to give

$$\log h = \frac{1}{1-\alpha} \left(\log c - \log(1-s) - \log \varpi - \alpha \log k \right).$$

The required elasticities are $\epsilon_{hk} = -\alpha/(1-\alpha)$ and $\epsilon_{h\varpi} = -1/(1-\alpha)$. Using these in (S18) and taking the term in dh/h from the right- to the left-hand side yields (41) as required. □