The Asymptotic Efficiency of Cointegration Estimators under Temporal Aggregation: Proofs of Lemmas, Theorems and Propositions

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APPENDIX A

Proof of Lemma A1. The most straightforward proof derives from taking the first m_1 equations from (4) which give

$$\Delta y_1(t) = -y_1(t-1) + By_2(t-1) + x_1(t),$$

which implies that $y_1(t) = By_2(t-1) + x_1(t)$. However, from (1), $y_1(t) = By_2(t) + u_1(t)$, and equating these two expressions yields

$$x_1(t) = u_1(t) + B\Delta y_2(t) = u_1(t) + Bx_2(t),$$

since the last m_2 equations of (4) give $\Delta y_2(t) = x_2(t)$. The last m_2 equations of (A3) yield, from inspection of the form of the matrices e^{-sJA} in the integral defining $\epsilon(t)$ in (A2), and J in (A3), that

$$x_2(t) = \int_0^1 u_2(t-s)ds,$$

which also uses the fact that $w_2(t) = u_2(t)$. The expression for $x_1(t)$ in the lemma now follows immediately. □

Proof of Lemma A2. Note that

$$y_2(t) = y_2(0) + \int_0^t u_2(s)ds,$$

so that integrating over $(t-1,t]$ yields

$$\int_0^1 y_2(t-s)ds = \int_{t-1}^t y_2(r)dr = y_2(0) + \int_{t-1}^t \int_0^r u_2(s)dsdr,$$

and hence

$$y_2(t) - \int_0^1 y_2(t-s)ds = \int_0^t u_2(s)ds - \int_{t-1}^t \int_0^r u_2(s)dsdr.$$

The double integral reduces to a pair of single integrals[1] as follows:

$$\begin{aligned} \int_{t-1}^t \int_0^r u_2(s)dsdr &= \int_{t-1}^t \int_0^{t-1} u_2(s)dsdr + \int_s^t \int_{t-1}^t u_2(s)dsdr \\ &= \int_0^{t-1} \left[\int_{t-1}^t dr\right] u_2(s)ds + \int_{t-1}^t \left[\int_s^t dr\right] u_2(s)ds \\ &= \int_0^{t-1} u_2(s)ds + \int_{t-1}^t (t-s)u_2(s)ds. \end{aligned}$$

Substituting this expression for the double integral gives

$$\begin{aligned} y_2(t) - \int_0^1 y_2(t-s)ds &= \int_0^t u_2(s)ds - \left[\int_0^{t-1} u_2(s)ds + \int_{t-1}^t (t-s)u_2(s)ds\right] \\ &= \int_{t-1}^t [1-(t-s)]u_2(s)ds = \int_0^1 (1-s)u_2(t-s)ds, \end{aligned}$$

as required. □

Proof of Lemma A3. The integral of interest is defined as the following limit in mean square:

$$\int_0^1 f(s)u(t-s)ds = \lim_{n\to\infty} E\left|\sum_{j=1}^n f_j u_j \ell(\Delta_j)\right|^2,$$

where $[0,1] = \cup_{j=1}^n \Delta_j$, $\ell(\Delta_j)$ denotes the Lebesgue measure (length) of each interval Δ_j, f_j denotes the value of $f(s)$ on Δ_j, and u_j is the value of $u(t-s)$ on Δ_j. Denote this mean square limit by $\sum_{j=1}^\infty f_j u_j \ell(\Delta_j)$. Then

$$\begin{aligned} \left[E\left|\int_0^1 f(s)u(t-s)ds\right|^\beta\right]^{1/\beta} &= \left[E\left|\sum_{j=1}^\infty f_j u_j \ell(\Delta_j)\right|^\beta\right]^{1/\beta} \\ &\leq \sum_{j=1}^\infty \left[E\left|f_j u_j \ell(\Delta_j)\right|^\beta\right]^{1/\beta} \quad \text{by Minkowski's inequality} \\ &\leq \sum_{j=1}^\infty |f_j| \left[E\left|u_j\right|^\beta\right]^{1/\beta} \ell(\Delta_j) \end{aligned}$$

[1] See Bergstrom (1997), McCrorie (1997) and Chambers (1999).

$$= \left[E\left|u_j\right|^{\beta}\right]^{1/\beta} \int_0^1 |f(s)|ds.$$

Raising both sides to the power β yields the required result. □

Proof of Lemma A4. Noting that $u(t-s) = L^s u(t) = e^{-sD}u(t)$ (see Priestley (1981, p.175) for details) it follows that the first integral has the representation

$$\int_0^1 u(t-s)ds = \left(\int_0^1 e^{-sD}ds\right) u(t) \equiv g(D)u(t),$$

where $g(z) = \int_0^1 e^{-sz}ds = (1-e^{-z})/z$. Proceeding in the same manner with the second integral,

$$\int_0^1 (1-s)u(t-s)ds = \left(\int_0^1 (1-s)e^{-sD}ds\right) u(t) \equiv k(D)u(t),$$

where $k(z) = \int_0^1 (1-s)e^{-sz}ds = \int_0^1 e^{-sz}ds - \int_0^1 se^{-sz}ds$. The first of these integrals is simply $g(z)$, while the second can be obtained by integration by parts (using the formula $\int_0^1 udv = [uv]_0^1 - \int_0^1 vdu$ with $u = s$ and $v = -e^{-sz}/z$):

$$\int_0^1 se^{-sz}ds = \left[-\frac{se^{-sz}}{z}\right]_0^1 + \int_0^1 \frac{e^{-sz}}{z}ds = \frac{1}{z}\left[g(z) - e^{-z}\right].$$

Hence $k(z) = g(z) - [g(z) - e^{-z}]/z$ which can be simplified further to give $k(z) = [1 - g(z)]/z$ as required. □

Proof of Lemma A5. By l'Hôpital's rule,

$$\lim_{z\to 0} g(z) = \lim_{z\to 0} \frac{1-e^{-z}}{z} = \lim_{z\to 0} e^{-z} = 1.$$

Expanding $h(z)$ gives $h(z) = [(z-1)e^{-z} + e^{-2z}]/z^2$ and application of l'Hôpital's rule gives

$$\lim_{z\to 0} h(z) = \lim_{z\to 0}\left(\frac{(2-z)e^{-z} - 2e^{-2z}}{2z}\right) = \lim_{z\to 0}\left(\frac{(z-3)e^{-z} + 4e^{-2z}}{2}\right) = \frac{1}{2}.$$

For integer $j \neq 0$, it is straightforward to show that $g(2\pi ij) = (1-e^{-2\pi ij})/(2\pi ij) = 0$ since $e^{-2\pi ij} = 1$ and that $h(2\pi ij) = e^{-2\pi ij}[1 - g(2\pi ij)]/(2\pi ij) = 1/(2\pi ij)$ since $e^{-2\pi ij} = 1$ and $g(2\pi ij) = 0$. □

Proof of Lemma A6. From the proof of Lemma A2,

$$\int_0^1 y_2(t-s)ds = y_2(0) + \int_0^{t-1} u_2(s)ds + \int_{t-1}^t (t-s)u_2(s)ds,$$

while

$$\begin{aligned}\frac{1}{p}\sum_{k=0}^{p-1} y_2\left(t-\frac{k}{p}\right) &= \frac{1}{p}\sum_{k=0}^{p-1}\left[y_2(0)+\int_0^{t-k/p} u_2(s)ds\right] \\ &= y_2(0)+\frac{1}{p}\sum_{k=0}^{p-1}\int_0^{t-k/p} u_2(s)ds \\ &= y_2(0)+\int_0^{t-1} u_2(s)ds+\frac{1}{p}\sum_{k=0}^{p-1}\int_{t-1}^{t-k/p} u_2(s)ds.\end{aligned}$$

Taking the difference of the two expressions above yields the first representation in the lemma. The filtering representation is obtained by noting first that

$$\int_0^1 su_2(t-s)ds = q(D)u_2(t),$$

where $q(z) = \int_0^1 se^{-sz}ds = [g(z) - e^{-z}]/z$; see the proof of Lemma A4. Secondly,

$$\frac{1}{p}\sum_{k=0}^{p-1}\int_{k/p}^{1} u_2(t-s)ds = r_p(D)u_2(t),$$

where

$$r_p(z) = \frac{1}{p}\sum_{k=0}^{p-1}\int_{k/p}^{1} e^{-sz}ds = \frac{1}{p}\sum_{k=0}^{p-1}\frac{1}{z}\left(e^{-(k/p)z}-e^{-z}\right) = \frac{1}{z}\left[\sigma_p(z)-e^{-z}\right].$$

The filtering representation follows since $m_p(z) = r_p(z) - q(z)$. □

Proof of Lemma A7. Since $h_p(z) = (e^{-z}\sigma_p(z)/z) - (e^{-z}g(z)/z)$, it is convenient to consider each term in turn. Since

$$\frac{e^{-z}}{z}\sigma_p(z) = \frac{e^{-z}}{z}\frac{1}{p}\sum_{k=0}^{p-1} e^{-(k/p)z} = \frac{1}{p}\sum_{k=0}^{p-1}\frac{e^{-(1+k/p)z}}{z},$$

application of l'Hôpital's rule yields

$$\begin{aligned}\lim_{z\to 0}\frac{e^{-z}}{z}\sigma_p(z) &= \lim_{z\to 0}\left(-\frac{1}{p}\sum_{k=0}^{p-1}\left(1+\frac{k}{p}\right)e^{-(1+k/p)z}\right) \\ &= -\frac{1}{p}\sum_{k=0}^{p-1}\left(1+\frac{k}{p}\right) = \frac{1}{2p}-\frac{3}{2}.\end{aligned}$$

Expanding the second term using the definition of $g(z)$ yields

$$\frac{e^{-z}}{z}g(z) = \frac{e^{-z}}{z}\left(\frac{1-e^{-z}}{z}\right) = \frac{e^{-z}-e^{-2z}}{z^2},$$

and application once more of l'Hôpital's rule yields

$$\lim_{z\to 0}\frac{e^{-z}}{z}g(z)=\lim_{z\to 0}\left(\frac{-e^{-z}+2e^{-2z}}{2z}\right)=\lim_{z\to 0}\left(\frac{e^{-z}-4e^{-2z}}{2}\right)=-\frac{3}{2}.$$

Combining these two limits gives $\lim_{z\to 0}h_p(z)=1/(2p)$. For integer $j\neq 0$,

$$h_p(2\pi ij)=\frac{e^{-2\pi ij}}{2\pi ij}\left[\sigma_p(2\pi ij)-g(2\pi ij)\right]=\frac{\sigma_p(2\pi ij)}{2\pi ij},$$

since $e^{-2\pi ij}=1$ and $g(2\pi ij)=0$ for integer $j\neq 0$. The properties of $\sigma_p(2\pi ij)$ yield the stated result. □

APPENDIX B

Proof of Lemma 1. Taking each sampling scheme in turn:

Scheme I: The expressions are obtained directly from partitioning (4), which gives $\xi_{1t}=x_1(t)$ and $\xi_{2t}=x_2(t)$, and then applying Lemma A1.

Scheme II: Taking the first m_1 equations of (4) gives

$$\Delta y_{1t}^S=-y_{1,t-1}^S+By_2^F(t-1)+x_1(t)=-y_{1,t-1}^S+By_{2,t-1}^F+\xi_{1t},$$

with $\xi_{1t}=x_1(t)+B[y_2^F(t-1)-y_{2,t-1}^F]$. The expression for ξ_{1t} in Lemma 1 results from the form of $x_1(t)$ given in Lemma A1 and using Lemma A2 for the expression

$$y_2^F(t-1)-y_{2,t-1}^F=y_2^F(t-1)-\int_0^1 y_2^F(t-1-s)ds.$$

The expression for ξ_{2t} is obtained from the last m_2 equations of (5), so that

$$\xi_{2t}=\int_0^1 x_2(t-s)ds.$$

Lemma A1 is again applied to relate ξ_{2t} to $u_2(t)$.

Scheme III: The first m_1 equations of (5) yield

$$\Delta y_{1t}^F=-y_{1,t-1}^F+B\int_0^1 y_2^S(t-1-s)ds+\int_0^1 x_1(t-s)ds=-y_{1,t-1}^F+By_{2,t-1}^S+\xi_{1t},$$

where ξ_{1t} is defined by

$$\xi_{1t}=\int_0^1 x_1(t-s)ds+B\left[\int_0^1 y_2^S(t-1-s)ds-y_{2,t-1}^S\right].$$

Application of Lemmas A1 and A2 yield the desired expression for ξ_{1t}, while the expression for ξ_{2t} is obtained directly from the last m_2 equations of (4) as in Scheme I.

Scheme IV: The expressions for ξ_{1t} and ξ_{2t} come directly from partitioning (5), so that

$$\xi_{1t} = \int_0^1 x_1(t-s)ds, \quad \xi_{2t} = \int_0^1 x_2(t-s)ds.$$

Lemma A1 once more yields the result. □

Proof of Lemma 2. The stationarity of ξ_t follows directly from the definitions of ξ_t in terms of $u(t)$ in Lemma 1. The invariance principle is justified by verifying that the conditions of Corollary 2.2 of Phillips and Durlauf (1986) are satisfied. The required conditions are that: (i) $E(\xi_t) = 0$; (ii) $E|\xi_{it}|^\beta < \infty$ for some $2 \leq \beta < \infty$; and (iii) the mixing conditions in Assumption 1(c) also hold for ξ_t. Condition (i) is satisfied by Assumptions 1(a) and Lemma 1. Condition (iii) holds due to the mixing properties in Assumption 1(c) and the fact that ξ_t is a measurable function of $u(t)$ involving finite lags, which follows from the definitions in Lemma 1 and application of Theorem 14.1 of Davidosn (1994) or Theorem 3.49 of White (1984). It remains to demonstrate that (ii) holds for each of the four sampling schemes.

Scheme I: From Lemma 1 it follows that:

$$\begin{aligned}
E|\xi_{1i,t}|^\beta &= E\left|u_{1i}(t) + \sum_{j=1}^{m_2} B_{ij}\int_0^1 u_{2j}(t-s)ds\right|^\beta \\
&\leq (m_2+1)^{\beta-1}\left[E|u_{1i}(t)|^\beta + \sum_{j=1}^{m_2} E\left|B_{ij}\int_0^1 u_{2j}(t-s)ds\right|^\beta\right] \\
&\leq (m_2+1)^{\beta-1}\left[E|u_{1i}(t)|^\beta + \sum_{j=1}^{m_2} |B_{ij}|^\beta E\left|\int_0^1 u_{2j}(t-s)ds\right|^\beta\right] \\
&\leq (m_2+1)^{\beta-1}\left[E|u_{1i}(t)|^\beta + \sum_{j=1}^{m_2} |B_{ij}|^\beta E\left|u_{2j}(t)\right|^\beta\right] < \infty
\end{aligned}$$

under Assumptions 1(b) and 2, and where the first inequality arises from the c_r-inequality and the third from application of Lemma A3. Also,

$$E|\xi_{2i,t}|^\beta = E\left|\int_0^1 u_{2i}(t-s)ds\right|^\beta \leq E\left|u_{2i}(t)\right|^\beta < \infty$$

by Assumption 1(b) and where Lemma A3 has been applied.

Scheme II: Applying a similar argument as in scheme I:

$$E|\xi_{1i,t}|^\beta = E\left|u_{1i}(t) + \sum_{j=1}^{m_2} B_{ij}\int_0^1 u_{2j}(t-s)ds + \sum_{j=1}^{m_2} B_{ij}\int_0^1 (1-s)u_{2j}(t-1-s)ds\right|^\beta$$

$$\leq (2m_2+1)^{\beta-1}\left[E|u_{1i}(t)|^\beta + \sum_{j=1}^{m_2}|B_{ij}|^\beta \left\{E\left|\int_0^1 u_{2j}(t-s)ds\right|^\beta + E\left|\int_0^1 (1-s)u_{2j}(t-1-s)ds\right|^\beta\right\}\right]$$

$$\leq (2m_2+1)^{\beta-1}\left[E|u_{1i}(t)|^\beta + \sum_{j=1}^{m_2}|B_{ij}|^\beta \left(E\left|u_{2j}(t)\right|^\beta + \left(\frac{1}{2}\right)^\beta E\left|u_{2j}(t)\right|^\beta\right)\right]$$

which is finite under the same conditions as scheme I and where $(1/2)^\beta = \left[\int_0^1 |1-s|ds\right]^\beta$. In a similar fashion,

$$\begin{aligned}
E|\xi_{2i,t}|^\beta &= E\left|\int_0^1 su_{2i}(t-s)ds + \int_0^1 (1-s)u_{2i}(t-1-s)ds\right|^\beta \\
&\leq (2)^{\beta-1}\left[E\left|\int_0^1 su_{2i}(t-s)ds\right|^\beta + E\left|\int_0^1 (1-s)u_{2j}(t-1-s)ds\right|^\beta\right] \\
&\leq (2)^{\beta-1}\left[\left(\frac{1}{2}\right)^\beta E|u_{2i}(t)|^\beta + \left(\frac{1}{2}\right)^\beta E|u_{2i}(t)|^\beta\right] < \infty,
\end{aligned}$$

due to Assumption 1(b) and since $(1/2)^\beta = \left[\int_0^1 |s|ds\right]^\beta$.

Scheme III: The proof for ξ_{1t} is similar to scheme I:

$$\begin{aligned}
E|\xi_{1i,t}|^\beta &= E\left|\int_0^1 u_{1i}(t-s)ds + \sum_{j=1}^{m_2} B_{ij}\int_0^1 su_{2j}(t-s)ds\right|^\beta \\
&\leq (m_2+1)^{\beta-1}\left[E\left|\int_0^1 u_{1i}(t-s)ds\right|^\beta + \sum_{j=1}^{m_2} E\left|B_{ij}\int_0^1 su_{2j}(t-s)ds\right|^\beta\right] \\
&\leq (m_2+1)^{\beta-1}\left[E|u_{1i}(t)|^\beta + \sum_{j=1}^{m_2}|B_{ij}|^\beta\left(\frac{1}{2}\right)^\beta E\left|u_{2j}(t)\right|^\beta\right] < \infty,
\end{aligned}$$

while $E|\xi_{2i,t}|^\beta < \infty$ identically as in scheme I.

Scheme IV: Following scheme II:

$$\begin{aligned}
E|\xi_{1i,t}|^\beta &= E\left|\int_0^1 u_{1i}(t-s)ds + \sum_{j=1}^{m_2} B_{ij}\int_0^1 su_{2j}(t-s)ds \right. \\
&\quad \left. + \sum_{j=1}^{m_2} B_{ij}\int_0^1 (1-s)u_{2j}(t-1-s)ds\right|^\beta \\
&\leq (2m_2+1)^{\beta-1}\left[E\left|\int_0^1 u_{1i}(t-s)ds\right|^\beta + \sum_{j=1}^{m_2}|B_{ij}|^\beta\left\{E\left|\int_0^1 su_{2j}(t-s)ds\right|^\beta \right.\right. \\
&\quad \left.\left. + E\left|\int_0^1 (1-s)u_{2j}(t-1-s)ds\right|^\beta\right\}\right]
\end{aligned}$$

$$\leq \quad (2m_2+1)^{\beta-1}\left[E|u_{1i}(t)|^{\beta}+2\sum_{j=1}^{m_2}|B_{ij}|^{\beta}\left(\frac{1}{2}\right)^{\beta}E\left|u_{2j}(t)\right|^{\beta}\right]<\infty$$

under Assumption 1(b). The demonstration that $E|\xi_{2i,t}|^{\beta}<\infty$ is the same as in scheme II and holds under the same conditions. □

Proof of Lemma 3. The filters are derived directly from the expressions in Lemma 1, which relate ξ_t to $u(t)$, along with the results in Lemma A4.

Scheme I: From Lemma 1 and application of Lemma A4,

$$\xi_{1t}=u_1(t)+B\int_0^1 u_2(t-s)ds=u_1(t)+Bg(D)u_2(t),$$

$$\xi_{2t}=\int_0^1 u_2(t-s)ds=g(D)u_2(t),$$

resulting in the form of $M(z)$ in the lemma.

Scheme II: The results in Lemma 1 and Lemma A4 give

$$\begin{aligned}\xi_{1t} &= u_1(t)+B\left[\int_0^1 u_2(t-s)ds+\int_0^1(1-s)u_2(t-1-s)ds\right]\\ &= u_1(t)+B[g(D)+h(D)]u_2(t),\end{aligned}$$

where $h(z)=e^{-z}k(z)$ and is defined in the lemma, while

$$\xi_{2t}=\int_0^1\int_0^1 u_2(t-r-s)drds=g(D)^2u_2(t),$$

as required.

Scheme III: Again from Lemmas 1 and A4,

$$\begin{aligned}\xi_{1t} &= \int_0^1 u_1(t-s)ds+B\int_0^1\int_0^1 u_2(t-r-s)drds-B\int_0^1(1-s)u_2(t-1-s)ds\\ &= g(D)u_1(t)+B[g(D)^2-h(D)]u_2(t),\end{aligned}$$

while

$$\xi_{2t}=\int_0^1 u_2(t-s)ds=g(D)u_2(t).$$

Scheme IV: From Lemmas 1 and A4,

$$\begin{aligned}\xi_{1t} &= \int_0^1 u_1(t-s)ds+B\int_0^1\int_0^1 u_2(t-r-s)drds=g(D)u_1(t)+Bg(D)^2u_2(t),\\ \xi_{2t} &= \int_0^1\int_0^1 u_2(t-r-s)drds=g(D)^2u_2(t),\end{aligned}$$

as required. $\square$

Proof of Theorem 1. In order to simplify the presentation of the required summations, the notation

$$S(X_1, f, X_2) = \sum_{j=-\infty}^{+\infty} X_1(2\pi ij) f(2\pi j) X_2(-2\pi ij),$$

is utilised, where $X_1(\cdot)$ (of dimension $l \times m$) and $X_2(\cdot)$ (of dimension $m \times n$) are matrix functions and $f(\cdot)$ (of dimension $m \times m$) represents a spectral density matrix or a submatrix thereof. In each of the four sampling schemes, the long run variance matrix of interest is given by $\Omega = 2\pi f_{\xi\xi}(0) = 2\pi S(M, f, M')$, where $M(z)$ is defined in Lemma 3. The submatrices of Ω yield $\Omega_{11.2} = \Omega_{11} - \Omega_{12}\Omega_{22}^{-1}\Omega_{21}$. Some of the summations involving the functions $g(z)$ and $h(z)$ simplify in view of Lemma A5. For example, $S(g, f, g) = f(0) = S(g^2, f, g^2)$, $S(g, f, h) = f(0)h(0) = (1/2)f(0)$ and $S(h, f, h) = (1/4)f(0) + \sum_{j\neq 0} |h(2\pi ij)|^2 f(2\pi ij)$.

Scheme I: From the definition of $M(z)$, the following submatrices of Ω are obtained:

$$\begin{aligned}
\Omega_{11} &= 2\pi[S(I_{m_1}, f_{11}, I_{m_1}) + BS(g, f_{21}, I_{m_1}) + S(I_{m_1}, f_{12}, g)B' + BS(g, f_{22}, g)B'] \\
&= 2\pi\left[\sum_{j=-\infty}^{\infty} f_{11}(2\pi j) + Bf_{21}(0) + f_{21}(0)'B' + Bf_{22}(0)B'\right]; \\
\Omega_{12} &= 2\pi[S(I_{m_1}, f_{12}, g) + BS(g, f_{22}, g)] = 2\pi[f_{12}(0) + Bf_{22}(0)]; \\
\Omega_{22} &= 2\pi S(g, f_{22}, g) = 2\pi f_{22}(0).
\end{aligned}$$

Combining these expressions gives

$$\Omega_{12}\Omega_{22}^{-1}\Omega_{21} = 2\pi[f_{12}(0)f_{22}(0)^{-1}f_{21}(0) + Bf_{21}(0) + f_{21}(0)'B' + Bf_{22}(0)B']$$

and hence, since $\Omega_{11.2} = \Omega_{11} - \Omega_{12}\Omega_{22}^{-1}\Omega_{21}$, it follows that

$$\begin{aligned}
\Omega_{11.2} &= 2\pi\left[\sum_{j\neq 0} f_{11}(2\pi j) + f_{11}(0) + Bf_{21}(0) + f_{21}(0)'B' + Bf_{22}(0)B'\right] \\
&\qquad - 2\pi\left[f_{12}(0)f_{22}(0)^{-1}f_{21}(0) + Bf_{21}(0) + f_{21}(0)'B' + Bf_{22}(0)B'\right] \\
&= 2\pi\left[\sum_{j\neq 0} f_{11}(2\pi j) + f_{11.2}(0)\right],
\end{aligned}$$

as given in the Theorem.

Scheme II: Here, the relevant matrices are given by

$$\begin{aligned}\Omega_{11} &= 2\pi[S(I_{m_1}, f_{11}, I_{m_1}) + BS(g+h, f_{21}, I_{m_1}) + S(I_{m_1}, f_{12}, g+h)B' \\ &\quad + BS(g+h, f_{22}, g+h)B'] \\ &= 2\pi\left[\sum_{j=-\infty}^{\infty} f_{11}(2\pi j) + Bf_{21}(0) + BS(h, f_{21}, I_{m_1}) + f_{12}(0)B' + S(I_{m_1}, f_{12}, h)B'\right] \\ &\quad + 2\pi[2Bf_{22}(0)B' + BS(h, f_{22}, h)B'];\end{aligned}$$

$$\Omega_{12} = 2\pi[S(I_{m_2}, f_{12}, g^2) + BS(g+h, f_{22}, g^2)] = 2\pi[f_{12}(0) + (3/2)Bf_{22}(0)];$$

$$\Omega_{22} = 2\pi S(g^2, f_{22}, g^2) = 2\pi f_{22}(0).$$

Combining these terms yields

$$\Omega_{12}\Omega_{22}^{-1}\Omega_{21} = 2\pi\left[f_{12}(0)f_{22}(0)^{-1}f_{21}(0) + \frac{3}{2}Bf_{21}(0) + \frac{3}{2}f_{12}(0)B' + \frac{9}{4}Bf_{22}(0)B'\right],$$

and hence

$$\begin{aligned}\Omega_{11.2} &= 2\pi\left[\sum_{j\neq 0} f_{11}(2\pi j) + f_{11}(0) + \frac{3}{2}Bf_{21}(0) + B\sum_{j\neq 0} h(2\pi ij)f_{21}(2\pi j) + \frac{3}{2}f_{12}(0)B'\right] \\ &\quad + 2\pi\left[\sum_{j\neq 0} f_{12}(2\pi j)h(-2\pi ij)B' + \frac{9}{4}Bf_{22}(0)B' + B\sum_{j\neq 0}|h(2\pi ij)|^2 f_{22}(2\pi j)B'\right] \\ &\quad - 2\pi\left[f_{12}(0)f_{22}(0)^{-1}f_{21}(0) + \frac{3}{2}Bf_{21}(0) + \frac{3}{2}f_{12}(0)B' + \frac{9}{4}Bf_{22}(0)B'\right] \\ &= 2\pi\left[\sum_{j\neq 0} f_{11}(2\pi j) + f_{11}(0) + B\sum_{j\neq 0} h(2\pi ij)f_{21}(2\pi j) + \sum_{j\neq 0} f_{12}(2\pi j)h(-2\pi ij)B'\right] \\ &\quad + 2\pi\left[B\sum_{j\neq 0}|h(2\pi ij)|^2 f_{22}(2\pi j)B' - f_{12}(0)f_{22}(0)^{-1}f_{21}(0)\right] \\ &= 2\pi\left[\sum_{j\neq 0} H(2\pi ij, B)f(2\pi j)H(2\pi ij, B)^* + f_{11.2}(0)\right],\end{aligned}$$

where $H(z, B)$ is defined in the text before the theorem.

Scheme III: In this case,

$$\Omega_{11} = 2\pi[S(g, f_{11}, g) + BS(g^2 - h, f_{21}, g) + S(g, f_{21}, g^2 - h)B']$$

$$+2\pi[BS(g^2 - h, f_{22}, g^2 - h)B']$$

$$= 2\pi\left[f_{11}(0) + \frac{1}{2}Bf_{21}(0) + \frac{1}{2}f_{12}(0)B' + \frac{1}{4}Bf_{22}(0)B'\right]$$

$$+2\pi\left[B\sum_{j\neq 0}|h(2\pi ij)|^2 f_{22}(2\pi j)B'\right];$$

$$\Omega_{12} = 2\pi[S(g, f_{12}, g) + BS(g^2 - h, f_{22}, g)] = 2\pi[f_{12}(0) + (1/2)Bf_{22}(0)];$$

$$\Omega_{22} = 2\pi S(g, f_{22}, g) = 2\pi f_{22}(0).$$

These expressions yield

$$\Omega_{12}\Omega_{22}^{-1}\Omega_{21} = 2\pi\left[f_{12}(0)f_{22}(0)^{-1}f_{21}(0) + \frac{1}{2}Bf_{21}(0) + \frac{1}{2}f_{12}(0)B' + \frac{1}{4}Bf_{22}(0)B'\right],$$

from which it follows that

$$\Omega_{11.2} = 2\pi\left[B\sum_{j\neq 0}|h(2\pi ij)|^2 f_{22}(2\pi j)B' + f_{11.2}(0)\right],$$

as required.

Scheme IV: In this case, the matrix filter is equal to $g(z)$ times the matrix filter $M(z)$ for scheme I, and so

$$\Omega_{11} = 2\pi[S(g, f_{11}, g) + BS(g^2, f_{21}, g) + S(g, f_{12}, g^2)B' + BS(g^2, f_{22}, g^2)B']$$

$$= 2\pi[f_{11}(0) + Bf_{21}(0) + f_{21}(0)'B' + Bf_{22}(0)B'];$$

$$\Omega_{12} = 2\pi[S(g, f_{12}, g^2) + BS(g^2, f_{22}, g^2)] = 2\pi[f_{12}(0) + Bf_{22}(0)];$$

$$\Omega_{22} = 2\pi S(g^2, f_{22}, g^2) = 2\pi f_{22}(0).$$

Combining these expressions gives

$$\Omega_{12}\Omega_{22}^{-1}\Omega_{21} = 2\pi[f_{12}(0)f_{22}(0)^{-1}f_{21}(0) + Bf_{21}(0) + f_{21}(0)'B' + Bf_{22}(0)B']$$

and hence

$$\Omega_{11.2} = 2\pi[f_{11}(0) + Bf_{21}(0) + f_{21}(0)'B' + Bf_{22}(0)B']$$

$$-2\pi\left[f_{12}(0)f_{22}(0)^{-1}f_{21}(0) + Bf_{21}(0) + f_{21}(0)'B' + Bf_{22}(0)B'\right]$$

$$= 2\pi f_{11.2}(0),$$

as required. $\square$

Proof of Proposition 1. The validity of Proposition 1 is established by considering, in turn, the three matrix differences $\Omega_{11.2}^{j} - \Omega_{11.2}^{IV}$ for $j = I, II, III$, and demonstrating that each one is positive semi-definite. The first of these differences is

$$\Omega_{11.2}^{I} - \Omega_{11.2}^{IV} = 2\pi \sum_{j\neq 0} f(2\pi j),$$

which is clearly (Hermitian) positive definite in view of $f(\lambda)$ being a spectral density matrix. Next,

$$\Omega_{11.2}^{II} - \Omega_{11.2}^{IV} = 2\pi \sum_{j\neq 0} H(2\pi ij, B) f(2\pi j) H(2\pi ij, B)^{*}.$$

Since $f(\lambda)$ is Hermitian positive definite, $H(2\pi ij)f(2\pi j)H(2\pi ij)^{*}$ is clearly positive semi-definite for each j, and hence so is the above sum over $j \neq 0$. Finally,

$$\Omega_{11.2}^{III} - \Omega_{11.2}^{IV} = 2\pi B \sum_{j\neq 0} |h(2\pi ij)|^{2} f_{22}(2\pi j) B'.$$

In view of Lemma A5, $|h(2\pi ij)|^{2} = (2\pi ij)^{-1}(-2\pi ij)^{-1} = (4\pi^{2}j^{2})^{-1} > 0$ for all $j \neq 0$, and since $f_{22}(\lambda)$ is Hermitian positive definite, the above sum is positive semi-definite. $\square$

Proof of Theorem 2. The proof follows straightforwardly from the definition of an optimal estimator by evaluating the long run variance matrix $\widetilde{\Omega} = 2\pi f_{ww}(0)$ and demonstrating that it is equal to $2\pi f(0)$. Since $w(t)dt = \Gamma(D)u(t)dt$, where

$$\Gamma(z) = \begin{bmatrix} (1+z)I_{m_1} & B \\ 0 & I_{m_2} \end{bmatrix},$$

it follows that $f_{ww}(\lambda) = \Gamma(i\lambda)f(\lambda)\Gamma(i\lambda)^{*}$ $(-\infty < \lambda < \infty)$. Evaluation of the sub-matrices of $f_{ww}(\lambda)$ yields

$$\widetilde{\Omega}_{11} = 2\pi f_{ww,11}(0) = 2\pi[f_{11}(0) + Bf_{21}(0) + f_{12}(0)B' + Bf_{22}(0)B'],$$

$$\widetilde{\Omega}_{12} = 2\pi f_{ww,12}(0) = 2\pi[f_{12}(0) + Bf_{22}(0)],$$

$$\widetilde{\Omega}_{22} = 2\pi f_{ww,22}(0) = 2\pi f_{22}(0).$$

The expression for $\widetilde{\Omega}_{11.2}$ follows straightforwardly. $\square$

Proof of Proposition 2. This follows directly from the fact that the variance matrices in the limiting distributions of the optimal etimators in Scheme IV (Theorem 1) and in the continuous record case (Theorem 2) are identical. □

Proof of Lemma 4. Considering each of the three sampling schemes in turn:

Scheme I(p): Here, $y_{t,p} = (1/p)\sum_{k=0}^{p-1} y(t-k/p)$ and so the discrete time ECM is obtained by applying the lag operator function $(1/p)\sum_{k=0}^{p-1} L^{(k/p)}$ to (4) yielding the ECM representation $\Delta y_{t,p} = -JAy_{t-1,p} + \xi_{t,p}$, where $\xi_{t,p} = (1/p)\sum_{k=0}^{p-1} x(t-k/p)$. Application of Lemma A1 for $x(t)$ yields the required expressions.

Scheme II(p): Taking the first m_1 equations of the equation in Scheme I(p) above gives

$$\begin{aligned}\Delta y_{1t,p}^S &= -y_{1,t-1,p}^S + B\frac{1}{p}\sum_{k=0}^{p-1} y_2^F\left(t-1-\frac{k}{p}\right) + \frac{1}{p}\sum_{k=0}^{p-1} x_1\left(t-\frac{k}{p}\right) \\ &= -y_{1,t-1,p}^S + By_{2,t-1}^F + \xi_{1t,p},\end{aligned}$$

where $\xi_{1t,p} = (1/p)\sum_{k=0}^{p-1} x_1(t-k/p) + B[(1/p)\sum_{k=0}^{p-1} y_2^F(t-1-k/p) - y_{2,t-1}^F]$. Applying Lemmas A1 and A6 yields the required expression. The vector $\xi_{2t,p}$ is obtained directly from the last m_2 equations of (5) and hence is the same as the expression in Scheme II of Lemma 1.

Scheme III(p): The first m_1 equations of (5) yield

$$\begin{aligned}\Delta y_{1t}^F &= -y_{1,t-1}^F + B\int_0^1 y_2^S(t-1-s)ds + \int_0^1 x_1(t-s)ds \\ &= -y_{1,t-1}^F + By_{2,t-1,p} + \xi_{1t,p},\end{aligned}$$

where $\xi_{1t,p} = \int_0^1 x_1(t-s)ds + B\left[\int_0^1 y_2^S(t-1-s)ds - y_{2,t-1,p}^S\right]$ and the result follows by using Lemmas A1 and A6. The expression for $\xi_{2t,p}$ comes from the last m_2 equations of the expression in Scheme I(p) above which describe the evolution of $y_{2t,p}^S$. □

Proof of Lemma 5. Taking each sampling scheme in turn:

Scheme I(p): Since $\xi_{t,p} = \sigma_p(D)\xi_t$, where ξ_t is from Scheme I in Lemma 1, it follows immediately that $\xi_{t,p} = \sigma_p(D)M(D)u(t)$, with $M(z)$ defined in Scheme I of Lemma 3, since $\xi_t = M(D)u(t)$.

Scheme II(p): From the definition of $\xi_{1t,p}$ it follows, using Lemma A6, that

$$\xi_{1t,p} = \sigma_p(D)u_1(t) + B\sigma_p(D)g(D)u_2(t) + Bm_p(D)e^{-D}u_2(t),$$

and since $h_p(z) = e^{-z}m_p(z)$, the filter on $u_2(t)$ is $\sigma_p(D)g(D) + h_p(D)$. The vector $\xi_{2t,p}$ is equal to ξ_{2t} in Scheme II, and so it follows immediately that $\xi_{2t,p} = g(D)^2u_2(t)$. The expression for $M_p(z)$ follows by combining these results.

Scheme III(p): Again from the definition of $\xi_{1t,p}$ and using Lemma A6,

$$\xi_{1t,p} = g(D)u_1(t) + Bg(D)^2u_2(t) - Be^{-D}m_p(D)u_2(t),$$

and the expression in the lemma follows by noting that $h_p(z) = e^{-z}m_p(z)$. Finally, the vector $\xi_{2t,p}$ is equivalent to $\xi_{2t,p}$ in Scheme I(p), and the expression for $M_p(z)$ then follows immediately. □

Proof of Theorem 3. The proof follows the same approach as the proof of Theorem 1, with $M_p(z)$ replacing $M(z)$. Expressions involving the filter function $\sigma_p(z)$ simplify in view of $\sigma_p(2\pi ij) = 1$ for $j = 0, \pm p, \pm 2p, \ldots$ and $\sigma_p(2\pi ij) = 0$ otherwise. For example, $S(\sigma_p, f, \sigma_p) = \sum_{j=-\infty}^{\infty} f(2\pi jp)$. An outline of the derivations is provided below.

Scheme I(p): The expressions of interest are:

$$\Omega_{11} = 2\pi[S(\sigma_p, f_{11}, \sigma_p) + Bf_{21}(0) + f_{21}(0)'B' + Bf_{22}(0)B'];$$

$$\Omega_{12} = 2\pi[S(\sigma_p g, f_{12}, \sigma_p g) + BS(\sigma_p g, f_{22}, \sigma_p g)] = 2\pi f_{12}(0) + Bf_{22}(0);$$

$$\Omega_{22} = 2\pi S(\sigma_p g, f_{22}, \sigma_p g) = 2\pi f_{22}(0).$$

Combining these expressions gives

$$\Omega_{12}\Omega_{22}^{-1}\Omega_{21} = 2\pi[f_{12}(0)f_{22}(0)^{-1}f_{21}(0) + Bf_{21}(0) + f_{21}(0)'B' + Bf_{22}(0)B']$$

and hence

$$\Omega_{11.2} = 2\pi[S(\sigma_p, f_{11}, \sigma_p) - f_{12}(0)f_{22}(0)^{-1}f_{21}(0)] = 2\pi\left[\sum_{j\neq 0} f_{11}(\pi jp) + f_{11.2}(0)\right],$$

since $S(\sigma_p, f_{11}, \sigma_p) = f_{11}(0) + \sum_{j\neq 0} f_{11}(2\pi jp)$.

Scheme II(p): Here, the relevant matrices are:

$$
\begin{aligned}
\Omega_{11} &= 2\pi[S(\sigma_p, f_{11}, \sigma_p) + BS((\sigma_p g + h_p), f_{21}, \sigma_p) + S(\sigma_p, f_{12}, (\sigma_p g + h_p))B' \\
&\quad + BS((\sigma_p g + h_p), f_{22}, (\sigma_p g + h_p))B'] \\
&= 2\pi \left[\sum_{j=-\infty}^{\infty} f_{11}(2\pi jp) + \left(1 + \frac{1}{2p}\right) B f_{21}(0) + B \sum_{j\neq 0} h_p(2\pi ijp) f_{21}(2\pi jp) \right] \\
&\quad + 2\pi \left[\left(1 + \frac{1}{2p}\right) f_{12}(0) B' + \sum_{j\neq 0} h_p(-2\pi ijp) f_{12}(2\pi jp) B' \right] \\
&\quad + 2\pi \left[\left(1 + \frac{1}{2p}\right)^2 B f_{22}(0) B' + B \sum_{j\neq 0} |h_p(2\pi ijp)|^2 f_{22}(2\pi jp) B' \right]; \\
\Omega_{12} &= 2\pi[S(\sigma_p, f_{12}, g^2) + BS((\sigma_p g + h_p), f_{22}, g^2)] \\
&= 2\pi\left[f_{12}(0) + \left(1 + \frac{1}{2p}\right) B f_{22}(0)\right]; \\
\Omega_{22} &= 2\pi S(g^2, f_{22}, g^2) = 2\pi f_{22}(0),
\end{aligned}
$$

which makes use of the properties of the filter function $h_p(z)$ given in Lemma A7. Combining these terms yields

$$
\begin{aligned}
\Omega_{12}\Omega_{22}^{-1}\Omega_{21} &= 2\pi \left[f_{12}(0) f_{22}(0)^{-1} f_{21}(0) + \left(1 + \frac{1}{2p}\right) B f_{21}(0) + \left(1 + \frac{1}{2p}\right) f_{21}(0)' B' \right] \\
&\quad + 2\pi \left[\left(1 + \frac{1}{2p}\right)^2 B f_{22}(0) B' \right],
\end{aligned}
$$

which, from the form of Ω_{11}, results in

$$
\begin{aligned}
\Omega_{11.2} &= 2\pi \left[\sum_{j\neq 0} f_{11}(2\pi jp) + B \sum_{j\neq 0} h_p(2\pi ijp) f_{21}(2\pi jp) + \sum_{j\neq 0} h_p(-2\pi ijp) f_{12}(2\pi jp) B' \right] \\
&\quad + 2\pi \left[B \sum_{j\neq 0} |h_p(2\pi ijp)|^2 f_{22}(2\pi jp) B' + f_{11.2}(0) \right] \\
&= 2\pi \left[\sum_{j\neq 0} H(2\pi ijp, B) f(2\pi jp) H(2\pi ijp, B)^* + f_{11.2}(0) \right],
\end{aligned}
$$

as required.

Scheme III(p): Here, the relevant matrices are:

$$
\begin{aligned}
\Omega_{11} &= 2\pi[S(g, f_{11}, g) + BS((g^2 - h_p), f_{21}, g) + S(g, f_{12}, (g^2 - h_p))B' \\
&\quad + BS((g^2 - h_p), f_{22}, (g^2 - h_p))B'] \\
&= 2\pi\left[f_{11}(0) + \left(1 - \frac{1}{2p}\right)Bf_{21}(0)\left(1 - \frac{1}{2p}\right)f_{12}(0)B'\left(1 - \frac{1}{2p}\right)^2 Bf_{22}(0)B'\right] \\
&\quad + 2\pi\left[B\textstyle\sum_{j\neq 0}|h_p(2\pi ij)|^2 f_{22}(2\pi j)B'\right]; \\
\Omega_{12} &= 2\pi[S(g, f_{12}, \sigma_p g) + BS((g^2 - h_p), f_{22}, \sigma_p g)] \\
&= 2\pi[f_{12}(0) + (\left(1 - \frac{1}{2p}\right)Bf_{22}(0)]; \\
\Omega_{22} &= 2\pi S(\sigma_p g, f_{22}, \sigma_p g) = 2\pi f_{22}(0),
\end{aligned}
$$

which makes use of the properties of the filter function $h_p(z)$ given in Lemma A7. Combining these terms yields

$$
\begin{aligned}
\Omega_{12}\Omega_{22}^{-1}\Omega_{21} \quad &= \quad 2\pi\left[f_{12}(0)f_{22}(0)^{-1}f_{21}(0) + \left(1 - \frac{1}{2p}\right)Bf_{21}(0) + \left(1 - \frac{1}{2p}\right)f_{21}(0)'B'\right] \\
&\quad + 2\pi\left[\left(1 - \frac{1}{2p}\right)^2 Bf_{22}(0)B'\right],
\end{aligned}
$$

which, from the form of Ω_{11}, results in

$$
\Omega_{11.2} = 2\pi\left[B\sum_{j\neq 0}|h(2\pi ijp)|^2 f_{22}(2\pi jp)B' + f_{11.2}(0)\right],
$$

as required. □

Proof of Proposition 3. As the expressions for the Ω_{22} matrices are identical in Theorems 1 and 3, the proof proceeds by comparing the $\Omega_{11.2}$ matrices defined in these theorems for the cases with unadjusted and adjusted data.

Scheme I(p): The expressions in the two theorems give:

$$
\begin{aligned}
\Omega_{11.2}^{I} - \Omega_{11.2}^{I(p)} &= 2\pi\left[\sum_{j\neq 0} f_{11}(2\pi j) + f_{11.2}(0)\right] - 2\pi\left[\sum_{j\neq 0} f_{11}(2\pi jp) + f_{11.2}(0)\right] \\
&= 2\pi\sum_{j\in J_p} f_{11}(2\pi j),
\end{aligned}
$$

where $J_p = \{j : j \in \mathcal{N}, j \neq 0, \pm p, \pm 2p, \ldots\}$ and $\mathcal{N} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ denotes the set of positive and negative integers, including zero. Since $f(\lambda)$ is Hermitian positive definite by assumption, it follows that this matrix difference is positive definite.

Scheme II(p): The appropriate difference is

$$\Omega^{II}_{11.2} - \Omega^{II(p)}_{11.2} = 2\pi \left[\sum_{j \neq 0} H(2\pi ij, B) f(2\pi j) H(2\pi ij, B)^* + f_{11.2}(0) \right]$$
$$-2\pi \left[\sum_{j \neq 0} H(2\pi ijp, B) f(2\pi jp) H(2\pi ijp, B)^* + f_{11.2}(0) \right]$$
$$= 2\pi \sum_{j \in J_p} H(2\pi ij, B) f(2\pi j) H(2\pi ij, B)^*,$$

which is positive definite in view of the form of $H(z, B)$ and the Hermitian positive definiteness of $f(\lambda)$.

Scheme III(p): The difference of interest is

$$\Omega^{III}_{11.2} - \Omega^{III(p)}_{11.2} = 2\pi \left[B \sum_{j \neq 0} |h(2\pi ij)|^2 f_{22}(2\pi j) B' + f_{11.2}(0) \right]$$
$$-2\pi \left[B \sum_{j \neq 0} |h(2\pi ijp)|^2 f_{22}(2\pi j) B' + f_{11.2}(0) \right]$$
$$= 2\pi B \sum_{j \in J_p} |h(2\pi ij)|^2 f_{22}(2\pi j) B',$$

which is positive definite since $|h(2\pi ij)|^2 = (4\pi^2 j^2)^{-1} > 0$ and $f(\lambda)$ is Hermitian positive definite. □

ADDITIONAL REFERENCES

Bergstrom, A. R. (1997): "Gaussian estimation of mixed-order continuous-time dynamic models with unobservable stochastic trends from mixed stock and flow data," *Econometric Theory*, 13, 467–505.

Chambers, M. J. (1999): "Discrete time representation of stationary and nonstationary continuous time systems," *Journal of Economic Dynamics and Control*, 23, 619–639.

McCrorie, J. R. (1997): "Deriving the exact discrete analog of a continuous time system," London School of Economics STICERD Discussion Paper No. EM/97/343.