The Purchasing Power Parity Puzzle, Temporal Aggregation, and Half-Life Estimation

Marcus J. CHAMBERS

University of Essex

February 2004

Supplementary results

Derivation of (3) and (4): First observe that $X_n = (1/M)\sum_{j=0}^{M-1} x_{Mn+j}$. Applying this operator to (1) yields

$$X_n = \beta\frac{1}{M}\sum_{j=0}^{M-1} x_{Mn+j-1} + \frac{1}{M}\sum_{j=0}^{M-1} u_{Mn+j}. \tag{A1}$$

Using backward substitution in (1) gives $x_t = \beta^{M-1}x_{t-M+1} + \sum_{k=0}^{M-2}\beta^k u_{t-k}$. Setting $t = Mn + j - 1$ in this expression and substituting the resulting expression into (A1) gives

$$\begin{aligned} X_n &= \beta\frac{1}{M}\sum_{j=0}^{M-1}\beta^{M-1}x_{Mn+j-M} + \beta\frac{1}{M}\sum_{j=0}^{M-1}\sum_{k=0}^{M-2}\beta^k u_{Mn+j-k-1} + \frac{1}{M}\sum_{j=0}^{M-1} u_{Mn+j} \\ &= \beta^M\frac{1}{M}\sum_{j=0}^{M-1} x_{Mn+j-M} + \frac{1}{M}\sum_{j=0}^{M-1}\left(u_{Mn+j} + \sum_{k=0}^{M-2}\beta^{k+1}u_{Mn+j-k-1}\right), \end{aligned}$$

which yields (3) and (4) directly. □

Derivation of (5) and (6): From (4) it may be shown that U_n has the representation

$$U_n = \frac{1}{M}\sum_{l=-(M-1)}^{M-1}\alpha_l u_{Mn+l} \quad \text{where} \quad \alpha_l = \begin{cases} \sum_{m=|l|}^{M-1}\beta^m, & l < 0, \\ \sum_{m=l}^{M-1}\beta^{M-1-m}, & l \geq 0. \end{cases} \tag{A2}$$

From (A2) and the white noise nature of u_t,

$$M^2E\left(U_n^2\right) = E\left(\sum_{k=-(M-1)}^{M-1}\alpha_k u_{Mn+k}\right)\left(\sum_{l=-(M-1)}^{M-1}\alpha_l u_{Mn+l}\right)$$

$$
\begin{aligned}
&= \sum_k \sum_l \alpha_k \alpha_l E\left(u_{Mn+k} u_{Mn+l}\right) \\
&= \sigma^2 \sum_l \alpha_l^2 \\
&= \sigma^2 \left[\sum_{l=0}^{M-1} \left(\sum_{m=l}^{M-1} \beta^{M-1-m} \right)^2 + \sum_{l=-(M-1)}^{-1} \left(\sum_{m=|l|}^{M-1} \beta^m \right)^2 \right] \\
&= \frac{\sigma^2}{(1-\beta)^2} \left[\sum_{l=0}^{M-1} \left(1-\beta^{M-l}\right)^2 + \sum_{l=1}^{M-1} \left(\beta^l - \beta^M\right)^2 \right] \\
&= \frac{\sigma^2}{(1-\beta)^2} \left[M\left(1+\beta^{2M}\right) - 2\beta \frac{(1-\beta^{2M})}{(1-\beta^2)} \right],
\end{aligned}
$$

the last equality following from some additional straightforward algebra.

Turning to ω_1, (A2) yields

$$
\begin{aligned}
M^2 E(U_{n-1}U_n) &= E\left(\sum_{k=-(M-1)}^{M-1} \alpha_k u_{Mn+k-M} \sum_{j=-(M-1)}^{M-1} \alpha_j u_{Mn+j} \right) \\
&= \sum_{j=-(M-1)}^{M-1} \sum_{k=-(M-1)}^{M-1} \alpha_j \alpha_k E(u_{Mn+j} u_{Mn+k-M}).
\end{aligned}
$$

Now $E(u_{Mn+j}u_{Mn+k-M}) = \sigma^2$ if $j = k - M$, $= 0$ otherwise, so it is only those values of $j = k - M$ that contribute to the expectation above. Hence $M^2E(U_{n-1}U_n) = \sigma^2 \sum_{k=1}^{M} \alpha_{k-M}\alpha_k$ and, substituting for α_{k-M} and α_k, this becomes

$$
\begin{aligned}
M^2 E(U_{n-1}U_n) &= \sigma^2 \sum_{k=1}^{M} \left(\sum_{m=|k-M|}^{M-1} \beta^m \right) \left(\sum_{n=k}^{M-1} \beta^{M-1-n} \right) \qquad \text{(A3)} \\
&= \frac{\sigma^2 \beta^M}{(1-\beta)^2} \sum_{k=1}^{M} \left(\beta^{-k} - 1\right)\left(1-\beta^{M-k}\right) \\
&= \frac{\sigma^2}{(1-\beta)^2} \frac{\left[\beta(1-\beta^{2M}) - M\beta^M(1-\beta^2)\right]}{(1-\beta^2)},
\end{aligned}
$$

the final equality following from some further, but straightforward, algebra. □

Derivation of (7): From the definition of $\widehat{B}$ it follows that

$$
\operatorname{plim} \widehat{B} = B + \frac{\operatorname{plim} N^{-1} \sum_n U_n X_{n-1}}{\operatorname{plim} N^{-1} \sum_n X_{n-1}^2} = B + \frac{E(U_n X_{n-1})}{E(X_{n-1}^2)},
$$

the expressions for the probability limits arising due to the stationary and ergodic nature of X_n under the assumptions of the model. Since X_n may be written $X_n = \sum_{j=0}^{\infty} B^j U_{n-j}$ it follows that $E(X_{n-1}U_n) = E\sum_j B^j U_{n-1-j} U_n = E(U_{n-1}U_n) = \omega_1$ because U_n is MA(1).

Turning to $E(X_{n-1}^2)$ it follows from (2) that

$$E\left(X_{n-1}^2\right) = E\left(\frac{1}{M}\sum_{j=0}^{M-1} x_{Mn+j-M}\right)^2 = M^{-2}\sum_{j=0}^{M-1}\sum_{k=0}^{M-1} E\left(x_{Mn+j-M}x_{Mn+k-M}\right).$$

Since, for some $v > 0$, $x_t = \beta^v x_{t-v} + \sum_{j=0}^{v-1}\beta^j u_{t-j}$, it follows that

$$E\left(x_t x_{t-v}\right) = \beta^v E\left(x_{t-v}^2\right) + \sum_{j=0}^{v-1}\beta^j E\left(u_{t-j}x_{t-v}\right) = \frac{\beta^v\sigma^2}{1-\beta^2}$$

in view of $E(u_{t-j}x_{t-v}) = 0$ for $j < v$ and $E(x_{t-v}^2) = \sigma^2/(1-\beta^2)$. Hence

$$M^2 E\left(X_{n-1}^2\right) = \frac{\sigma^2}{1-\beta^2}\sum_{j=0}^{M-1}\sum_{k=0}^{M-1}\beta^{|j-k|} = \frac{\sigma^2}{1-\beta^2}\left[\frac{M(1-\beta^2) - 2\beta(1-\beta^M)}{(1-\beta)^2}\right]. \tag{A4}$$

Combining (A3) and (A4) yields (7) as required. $\square$